Gauge theories from quantum strings

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The AdS/CFT correspondence relates a gravitational theory (a theory of closed strings) to a gauge theory with no gravity at all

Maldacena '97

The correspondence offers a spectacular new insight into

- dynamics of strongly coupled gauge fields
- black holes
- many-body physics

A dream:

Find a string description of realistic confining theories

The fundamental model of AdS/CFT:

$$\mathcal{N} = 4 \text{ super Yang} - \text{Mills} \quad \Leftrightarrow \quad \text{closed strings in AdS}_5 \times \text{S}^5 \text{ geometry}$$

Research on the fundamental model of AdS/CFT

- Initial research was concentrated on deriving gauge theory correlators from supergravity

  Gubser, Klebanov and Polyakov '98

  Witten '98
- Studies of unprotected operators with large R-charge

  Berenstein, Maldacena and Nastase '02
- Discovery of integrable structures in gauge and string theory

In spite of important recent progress, the exact spectra of both $\mathcal{N} = 4$ super Yang-Mills and strings on $\mathrm{AdS}_5 \times \mathrm{S}^5$ remain unknown

My goal is to explain the progress towards solving the spectral problem of the fundamental model based on the ideas of exact integrability

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Outline

N=4 super Yang-Mills theory

- Maximally supersymmetric field theory in 4dim:

$$A_\mu, \quad \Phi^i, \quad i = 1, \dots, 6 \quad \text{and} \quad 4 \text{ Weyl fermions}$$

  all fields in the adjoint of $U(N)$.

- Lagrangian

$$\mathscr{L} = \frac{1}{g_{\rm YM}^2} \mathrm{Tr}\Big[\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \tfrac{1}{2} D_\mu \Phi^i D^\mu \Phi^i - \tfrac{1}{4} [\Phi^i, \Phi^j]^2 + \text{fermions}\Big]$$

- It is an exact (super) conformal theory in four dimensions
- Conformal symmetry includes Poincaré algebra, dilatation and conformal boosts
- g_{YM} is not running; it is merely a parameter. Another parameter is the rank N of the gauge group

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Conformal theories – CFT's

- CFT is characterized by a set of *primary* operators $\{\mathcal{O}_i\}$. Primary operators correspond to eigenstates of the dilatation

$$D \cdot \mathcal{O} = i\Delta\, \mathcal{O}$$

Δ is the scaling dimension

- A CFT is described by 2- and 3-point cor. functions of $\mathcal{O}$

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(y)\rangle = \frac{\delta_{ij}}{|x-y|^{2\Delta_i}}$$

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(y)\mathcal{O}_k(z)\rangle = \frac{C_{ijk}}{|x-y|^{\Delta_i+\Delta_j-\Delta_k}|x-z|^{\Delta_i+\Delta_k-\Delta_j}|y-z|^{\Delta_j+\Delta_k-\Delta_i}}$$

- Composite gauge invariant operators $\Leftrightarrow$ 'observables'

$$\mathcal{O} = \mathrm{Tr}\Big[\ldots F_{\mu\nu} D_\rho \Phi^i \ldots \Psi^k D_\lambda \Phi^j \Phi^m \ldots\Big]$$

gauge-invariant products of elementary fields

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Scaling dimensions

- The composite operators

$$\mathscr{O} = \mathrm{Tr}\Big[\dots F_{\mu\nu} D_\rho \Phi^i \dots \Psi^k D_\lambda \Phi^j \Phi^m \dots\Big]$$

  mix under renormalization

$$\langle \mathscr{O}_i(x) \mathscr{O}_j(y) \rangle = \frac{1}{|x-y|^{2\Delta_{\mathrm{class}}}} \Big[\delta_{ij} + \lambda\, M_{ij}\, \log \Lambda + \cdots\Big]$$

  where $\lambda = g_{\mathrm{YM}}^2 N$ is the 't Hooft coupling

- Diagonalization of the mixing matrix M leads to the appearance of the "anomalous" dimension:

$$\Delta_{\mathrm{class}} \quad \Rightarrow \quad \Delta(g_{\mathrm{YM}}, 1/N) \equiv \Delta(\lambda, 1/N)$$

- Mixing problem simplifies in the limit $N \to \infty$, where a wonderful connection to integrable models and string theory emerges!

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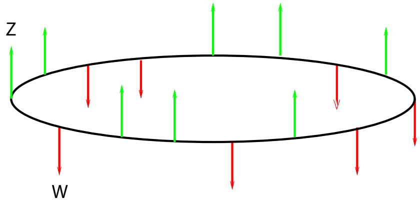
Planar scaling dimensions via integrable spin chains

$$\mathcal{O} = \mathrm{tr}(Z^{L-M} W^M) \;, \quad Z = \Phi^1 + i\Phi^2 \,, \quad W = \Phi^3 + i\Phi^4$$



A closed spin chain of length L

Planar scaling dimensions via integrable spin chains

The Hamiltonian H acts as $2^L \times 2^L$ matrix, where L is the length of the chain. *M* is a number of magnons

At one loop the Hamiltonian of the $\mathfrak{su}(2)$ spin chain is

$$\mathbf{H} = \sum_{i=1}^{L} \left(I - P_{i,i+1} \right) , \qquad P(\uparrow\downarrow) = (\downarrow\uparrow)$$

The Heisenberg spin chain – paradigmatic integrable model of condensed matter physics. Solved by the Bethe ansatz.

Minahan and Zarembo, '03

Previously observed integrable structures in QCD: Lipatov, '94; Faddeev and Korchemsky '95

Higher-loop integrability

Conformal Hamiltonian H defines an integrable long-range spin chain

$$\mathbf{H}_{1\ell} = \sum_{i=1}^{L} \left(I - P_{i,i+1} \right) \quad \Leftarrow \quad \text{Heisenberg Hamiltonian}$$

$$\mathbf{H}_{2\ell} = \sum_{i=1}^{L} \left(-\frac{3}{2} I + 2P_{i,i+1} - \frac{1}{2} P_{i,i+2} \right)$$

$$\mathbf{H}_{3\ell} = \sum_{i=1}^{L} \Big(5I - 7P_{i,i+1} + 2P_{i,i+2} - \frac{1}{2}(P_{i,i+3}P_{i+1,i+2} - P_{i,i+2}P_{i+1,i+3}) \Big)$$

Beisert, Kristjansen and Staudacher '03

Integrability:

- Elementary excitations are *magnons* (quasi-particles with momenta p_k)
- Existence of family of commuting charges $\{Q_i\}$: $[\mathbf{H}, Q_i(\lambda)] = [Q_i(\lambda), Q_j(\lambda)] = 0$ $\Rightarrow$ elastic scattering
- In the limit $L \rightarrow \infty$ the Hamiltonian can be diagonalized by *the Bethe Ansatz*

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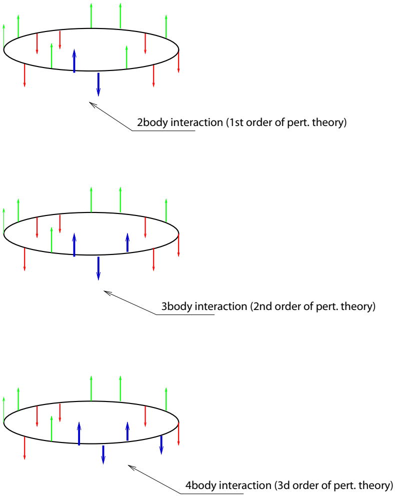
Obscuring spin chains at higher loops



At higher orders of perturbation theory interactions “wrap” around the circle making the spin chain interpretation obscure

The planar AdS/CFT duality conjecture

- Planar scaling dimensions $\Delta(\lambda)$ in Yang-Mills theory should be computable by string theory! Simultaneously, this should test the conjecture.

- The string theory is type IIB superstring moving in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ space-time

- The action for $X^M(\tau,\sigma), \quad M = 1, \ldots, 10$

$$S = -\frac{g}{2}\int d\tau d\sigma \sqrt{-h} h^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N G_{MN}(X) + \text{fermions}$$

Metsaev, Tseytlin '98

Strings are closed $\Leftrightarrow$ sigma-model is defined on a cylinder

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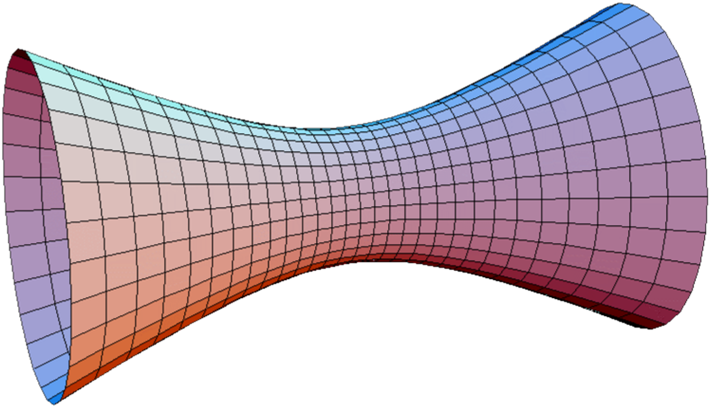
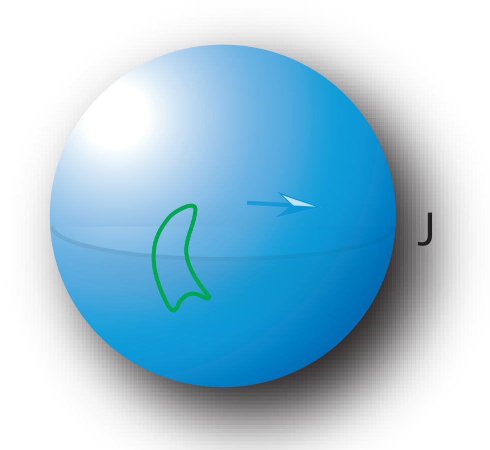
Anti-de Sitter space
Maximally symmetric space of constant negative curvature



String energy E is a conserved Noether charge corresponding to the $\mathrm{SO}(2)$ subgroup of the conformal group $\mathrm{SO}(4,2)$



J is a conserved Noether charge corresponding to one of the Cartan generators of $SO(6)$

The planar AdS/CFT duality conjecture

The conformal+R-symmetry groups $\mathrm{SO}(4,2) \times \mathrm{SO}(6)$

- Symmetry group of the $\mathcal{N} = 4$ super Yang-Mills
- Isometry group of $\mathrm{AdS}_5 \times \mathrm{S}^5$ space-time, i.e. the global symmetry group of string sigma model
- Representations are described by a set of numbers

$$[\Delta = E, S_1, S_2; J_1, J_2, J_3]$$

i.e. by energy, spins and angular momenta. Apparently only Δ can continuously depend on g.

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- The gauge-string correspondence

't Hooft coupling λ	$\Leftrightarrow$	Inverse string tension $g = \frac{\sqrt{\lambda}}{2\pi}$
SYM operators	$\Leftrightarrow$	String states
Scaling dimension $\Delta(\lambda)$	$\Leftrightarrow$	String energy $E(g)$

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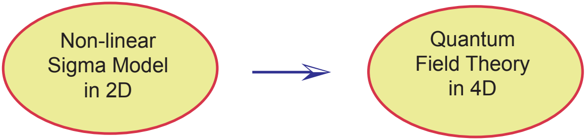

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- To compute $E(g)$ and therefore $\Delta(g)$, one needs to solve the 2-dim quantum sigma model on a cylinder! Very hard ...
- In the last 5 years a lot of evidence has been collected that string integrability is the key to the solution

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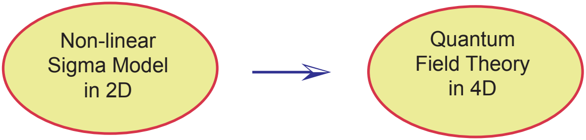

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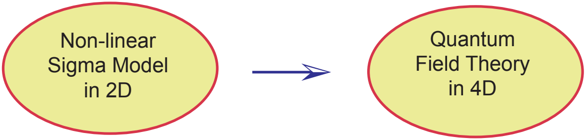

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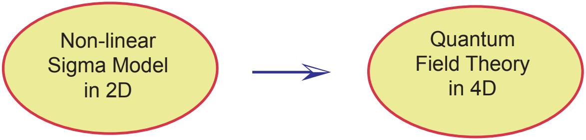
The planar AdS/CFT duality conjecture



$$S = -\frac{g}{2}\int d\tau d\sigma \sqrt{-h}h^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N G_{MN}(X) + \text{fermions}$$

- To compute $E(g)$ and therefore $\Delta(g)$, one needs to solve the 2-dim quantum sigma model on a cylinder! Very hard ...
- In the last 5 years a lot of evidence has been collected that string integrability is the key to the solution

$AdS_5 \times S^5$ superstring in the light-cone gauge

- Classical string sigma model is integrable: it exhibits an infinite number of conservation laws! Bena, Polchinski and Roiban '03
- Quantum integrability is a plausible assumption!
- Sigma model has a local diffeomorphism symmetry. It is eliminated through the light-cone gauge fixing. Frolov and G.A. '04
- Sigma model is on a cylinder of circumference $P_+ = J$, where J is an angular momentum of string around S^5
- Sigma model has soliton solutions – "giant magnons"

Hofman and Maldacena '06

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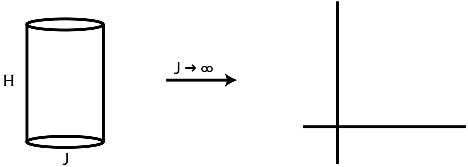
Integrability on a plane $\Leftrightarrow$ Factorized Scattering

- When $J \to \infty$ the cylinder decompactifies into a plane



- Integrability implies:
  - the number of particles is conserved
  - scattering permutes momenta
  - any multi-particle scattering process is factorised into a sequence of two-body events. Two-particle S-matrix $S(p_1, p_2)$ is the main object

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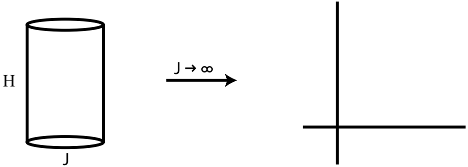

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Dispersion relation and the two-body S-matrix

- Particles form a 16-dim multiplet of l.c. symmetry algebra
- Exact dispersion relation for string excitations

$$\epsilon(p) = \sqrt{1 + 4g^2 \sin^2 \frac{p}{2}}$$

Beisert, Dippel and Staudacher '04

- Exact two-body S-matrix

$$S_{256\times 256}(p_1, p_2) \quad \Leftarrow \quad \text{exact in } g$$

was found from various symmetry considerations and the perturbative data

Frolov, Staudacher and G.A. '04; Staudacher '04; Beisert '05; Frolov, Zamaklar and G.A. '06 Janik '06; Beisert, Hernandez and Lopez '06 Beisert, Eden and Staudacher '06

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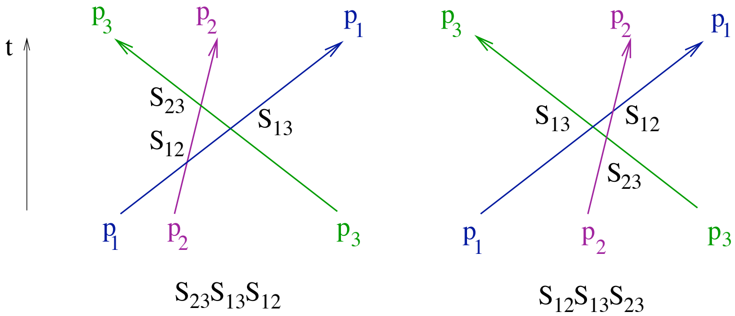
Properties of the S-matrix

- $S_{23}S_{13}S_{12} = S_{12}S_{13}S_{23}$ $\longleftarrow$ Yang-Baxter equation
- $S_{12}(p_1^*, p_2^*)\, S_{12}(p_1, p_2)^\dagger = \mathbb{I}$ $\longleftarrow$ generalized physical unitarity

- $S_{12}(p_1, p_2)^T = I_{12}^g S_{12}(p_1, p_2) I_{12}^g$ $\longleftarrow$ CPT invariance
- $S_{12}(p_1, p_2)^{-1} = S_{12}(-p_1, -p_2)$ $\longleftarrow$ parity transformation
- $S_{12}(p_1, p_2) S_{21}(p_2, p_1) = \mathbb{I}$ $\longleftarrow$ unitarity
- $S_{21}(p_2^*, p_1^*) = S_{12}(p_1, p_2)^\dagger$ $\longleftarrow$ hermitian analyticity
- $\mathscr{C}_1^{-1} \mathcal{S}_{12}^{t_1}(p_1, p_2) \mathscr{C}_1 \mathcal{S}_{12}(-p_1, p_2) = \mathbb{I}$ $\longleftarrow$ crossing

Factorized scattering



Spectrum on a large circle

- Bethe-Yang equations

$$\text{“}e^{ip_k J}\prod_{k\neq i}^{M} S(p_i,p_k)=1\text{”}$$

(+ additional equations with auxiliary roots encoding non-diagonal structure of $\mathcal{S}$)

Beisert,Staudacher ’04

- Given $\{p_i\}_{i=1}^{M}$, the energy (dimension) is given by

$$E=\sum_{i=1}^{M}\epsilon(p_i)=\sum_{i=1}^{M}\sqrt{1+4g^2\sin^2\frac{p_i}{2}}=E(g,J)$$

- This is incorrect answer for finite J!

Higher loop Feynman graphs, finite-size corrections to classical string energies, etc., all points to this...

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TBA and mirror theory

Follow the TBA approach for relativistic models (Zamolodchikov '90)

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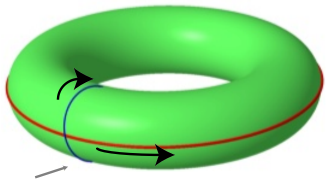

- One Euclidean theory – two Minkowski theories. One is related to the other by the double Wick rotation:

$$\tilde{\sigma} = -i\tau \,, \qquad \tilde{\tau} = i\sigma$$

The Hamiltonian $\tilde{\mathrm{H}}$ w.r.t. $\tilde{\tau}$ defines the *mirror theory*.

- Ground state energy ($R \to \infty$) is related to the free energy of its mirror

$$E(L) = Lf(L)$$

Free energy f can be found from the Bethe ansatz for the mirror model

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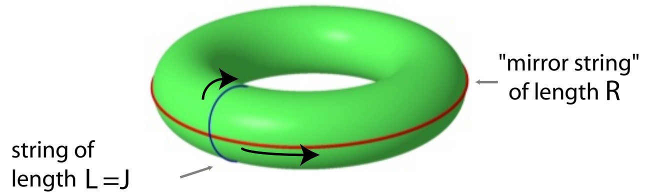

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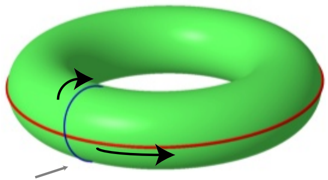

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Mirror dispersion relation

- The pole of the Euclidean two-point function

$$\mathrm{H}_{\mathrm{E}}^2 + 4g^2 \sin^2 \frac{p_{\mathrm{E}}}{2} + 1$$

- In string theory: $\mathrm{H}_{\mathrm{E}} \to -i\mathrm{H}, \quad p_{\mathrm{E}} \to p \quad \Longrightarrow$

$$\mathrm{H} = \sqrt{1 + 4g^2 \sin^2 \frac{p}{2}}$$

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- Magnitude of the correction ($L \equiv J$) at weak coupling

$$magnitude \sim e^{-L\tilde{\mathrm{H}}} = e^{-2J\mathrm{arcsinh}\frac{\sqrt{1+\tilde{p}^2}}{2g}} \overset{g\to 0}{\sim} \frac{g^{2J}}{(1+\tilde{p}^2)^J} + \ldots\ldots$$

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TBA equations for pseudo-energies of mirror particles

- Q-particles
$$\epsilon_Q = L\widetilde{\mathcal{E}}_Q - \log\left(1+e^{-\epsilon_{Q'}}\right)\star K_{\mathfrak{sl}(2)}^{Q'Q} - \log\left(1+e^{-\epsilon^{(\alpha)}_{M'|vw}}\right)\star K^{M'Q}_{vwx}$$
$$-\log\left(1-e^{ih_\alpha-\epsilon^{(\alpha)}_{y-}}\right)\star K^{yQ}_{-} - \log\left(1-e^{ih_\alpha-\epsilon^{(\alpha)}_{y+}}\right)\star K^{yQ}_{+}$$
- y-particles
$$\epsilon^{(\alpha)}_{y\pm} = -\log\left(1+e^{-\epsilon_Q}\right)\star K^{Qy}_{\pm} + \log\frac{1+e^{-\epsilon^{(\alpha)}_{M|vw}}}{1+e^{-\epsilon^{(\alpha)}_{M|w}}}\star K_M$$
- $M|vw$-strings
$$\epsilon^{(\alpha)}_{M|vw} = -\log\left(1+e^{-\epsilon_{Q'}}\right)\star K^{Q'M}_{xv}$$
$$+\log\left(1+e^{-\epsilon^{(\alpha)}_{M'|vw}}\right)\star K_{M'M} - \log\frac{1-e^{ih_\alpha-\epsilon^{(\alpha)}_{y+}}}{1-e^{ih_\alpha-\epsilon^{(\alpha)}_{y-}}}\star K_M$$
- $M|w$-strings
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- The ground state energy
$$E(L) = -\int \mathrm{d}u\, \sum_{Q=1}^{\infty} \frac{1}{2\pi}\frac{d\widetilde{p}^Q}{du}\log\left(1+e^{-\epsilon_Q}\right)$$

Frolov and G.A. '09

See also Bombardelli, Fioravanti, Tateo '09; Gromov, Kazakov, Kozak and Vieira '09

Infinite system of coupled equations. Analysis is underway.

Konishi operator in perturbation theory

- Konishi operator is the simplest non-protected operator in $\mathcal{N} = 4$ SYM:

$$\mathrm{Tr}\, \Phi_i^2$$

- It has a susy descendent

$$\mathrm{Tr}(W^2 Z^2) \quad \rightarrow \quad J = 2$$

- Solving BY equations iteratively for $M = 2$, one finds $p_1 = -p_2 = p$ with

$$p = \frac{2\pi}{3} - \sqrt{3}g^2 + \frac{9\sqrt{3}}{2}g^4 - \frac{72\sqrt{3} + 8 \cdot 8\sqrt{3}\zeta(3)}{3}g^6 + \ldots$$

- This gives the energy

$$E_{\mathrm{BY}} = \underbrace{4 + 12g^2 - 48g^4 + 336g^6}_{\text{agrees with pert.compt}} - (2820 + 288\zeta(3))g^8 + \ldots$$

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Fiamberti, Santambrogio, Sieg , Zanon '07 ($\sim$ 200 supergraphs!)
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- Compare to the result based on BY equations:

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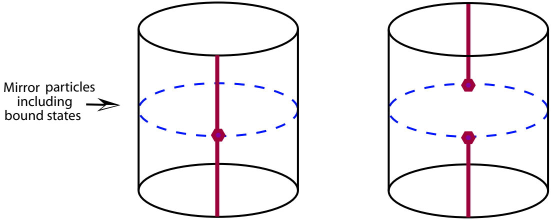

Lüscher corrections: the F- and μ-terms

In relativistic QFT's the leading correction to single particle energies is due to Lüscher

$$E_n(L) = m\cosh\theta_n - \underbrace{m\int_{-\infty}^{+\infty}\frac{d\theta}{2\pi}\frac{\cosh(\theta-\theta_n)}{\cosh\theta_n}\Big(S(\theta+\frac{i\pi}{2}-\theta_n)-1\Big)e^{-mL\cosh\theta}}_{\text{F}-\text{term}}$$

$$+ \underbrace{\text{residues}}_{\mu-\text{term}}$$

Konishi operator in perturbation theory

The leading exponential correction in L was found by BJ by generalizing the Lüscher formulae

- to multi-particle states
- to a non-Lorentz invariant case
- to non-diagonal scattering

$$\Delta E_n = E_n(L) - E_n^{\mathrm{BY}}(L) =$$
$$- \sum_Q \int \frac{d\tilde{p}}{2\pi} \sum_{Q_1,\ldots,Q_n} (-1)^F [S_{Q_1 a}^{Q_2 a}(\tilde{p}, p_1) S_{Q_2 a}^{Q_3 a}(\tilde{p}, p_2) \ldots S_{Q_n a}^{Q_1 a}(\tilde{p}, p_n)] e^{-\tilde{H}_a(\tilde{p})L}$$

- $p_1, \ldots, p_n$ are momenta of physical particles in string theory
- $\tilde{p}$ is the momentum of a Q-particle in the mirror theory
- The leading large L approx. to the exact TBA should reproduce this formula

For the relativistic $O(4)$, see Gromov, Kazakov, Vieira '08

Konishi operator in perturbation theory

The leading exponential correction in L was found by BJ by generalizing the Lüscher formulae

- to multi-particle states
- to a non-Lorentz invariant case
- to non-diagonal scattering

$$\Delta E_n = E_n(L) - E_n^{\rm BY}(L) = - \sum_Q \int \frac{d\tilde{p}}{2\pi} \sum_{Q_1,\ldots,Q_n} (-1)^F [S_{Q_1 a}^{Q_2 a}(\tilde{p}, p_1) S_{Q_2 a}^{Q_3 a}(\tilde{p}, p_2) \ldots S_{Q_n a}^{Q_1 a}(\tilde{p}, p_n)] e^{-\tilde{H}_a(\tilde{p})L}$$

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$$\begin{aligned}\Delta E_n &= E_n(L) - E_n^{\mathrm{BY}}(L) = \\ &- \sum_Q \int \frac{d\tilde{p}}{2\pi} \sum_{Q_1,\ldots,Q_n} (-1)^F [S_{Q_1 a}^{Q_2 a}(\tilde{p}, p_1) S_{Q_2 a}^{Q_3 a}(\tilde{p}, p_2) \ldots S_{Q_n a}^{Q_1 a}(\tilde{p}, p_n)] e^{-\tilde{H}_a(\tilde{p})L}\end{aligned}$$

- $p_1, \ldots, p_n$ are momenta of physical particles in string theory
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Conclusions

The spectral problem for $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring in the light-cone gauge $P_+ = J$:

- *Infinite J spectrum is trivial*
- *Large but finite J spectrum is encoded in the BY equations based on the known exact S-matrix. Corrections exponential in J are missed*
- *Finite J spectrum is encoded into an infinite set of coupled TBA equations in the mirror theory*
- *Lüscher correction perfectly reproduces the direct perturbative result which goes beyond the validity of the BY equations. Highly non-trivial check of the mirror theory approach*