Dynamic $\mathcal{N} = 4$ Superconformal Symmetry*

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Overview:

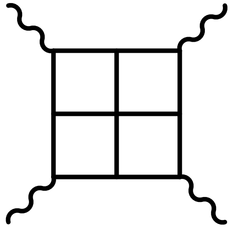
Scattering Amplitudes and Dual Conformal Symmetry

Planar Scattering Amplitudes

Intriguing result in $\mathcal{N}=4$ SYM in the planar limit $N_\mathrm{c}\to\infty$:

Four-gluon scattering amplitude obeys BDS relation [Anastasiou, Bern, Dixon, Kosower] [Bern, Dixon, Smirnov]

$$A(p,\lambda)\simeq A^{(0)}(p)\exp\Big(2D_\mathrm{cusp}(\lambda)M^{(1)}(p)+F(p,\lambda)\Big).$$



Only required data: • tree level, • one loop, • cusp dimension.

- No finite remainder function $F(p,\lambda)=0$.
- Captures correct IR singularities.

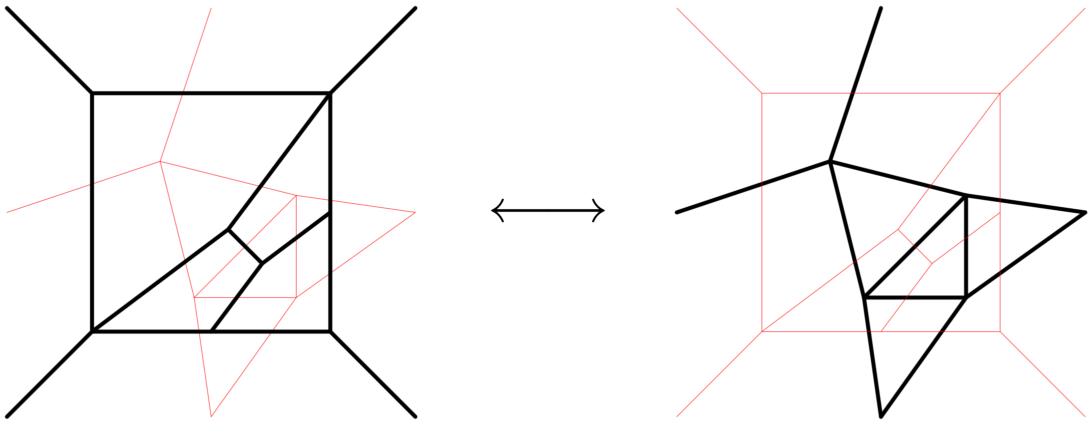
Scattering amplitudes constructible by unitarity and suitable ansatz.
Verified BDS relation at $\mathcal{O}(\lambda^4)$ with [Bern, Dixon, Smirnov] [Bern, Czakon, Dixon, Kosower, Smirnov]

$$D_\mathrm{cusp}(\lambda)=\frac{1}{2}\frac{\lambda}{\pi^2}-\frac{1}{96}\frac{\lambda^2}{\pi^2}+\frac{11}{23040}\frac{\lambda^3}{\pi^2}-\left(\frac{73}{2580480}+\frac{\zeta(3)^2}{1024\pi^6}\right)\frac{\lambda^4}{\pi^2}\pm\ldots.$$

Simplicity and Dual Conformal Symmetry

Planar amplitudes and integrals simpler than expected:

- No bubbles, no triangles, typically scalar boxes.
- Similarity of momentum and position space propagators in $D = 4$.



- Dual amplitudes and integrals are conformal. [Drummond Korchemsky Sokatchev] . . .
- Dual superconformal symmetry of tree amplitudes. [Drummond, Henn Korchemsky Sokatchev]

Strong coupling:

- Free strings self-dual under supersymmetric T-duality. [Alday Maldacena] [Berkovits Maldacena]
- Dual superconformal symmetry $\widehat{=}$ symmetry of T-dual model.

Finite Remainder

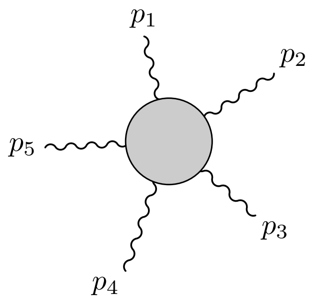
Dual superconformal symmetry allows $F(p,\lambda)$ only for $n \geq 6$ legs

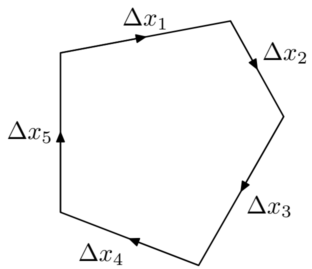
$$A(p,\lambda) \simeq A^{(0)}(p) \exp\Big(2D_{\text{cusp}}(\lambda) M^{(1)}(p) + F(p,\lambda)\Big).$$

Indications for $F(p,\lambda) \neq 0$ in general case:

- Unitarity construction for six legs, two loops. [Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich]
- Further indications. [Drummond, Henn, Korchemsky, Sokatchev] [Bartels, Lipatov, Sabio Vera]

Also from duality to light-like Wilson loops. [Alday, Maldacena] [Drummond, Korchemsky, Sokatchev] [Alday, Maldacena] [Drummond, Henn, Korchemsky, Sokatchev]





⋆ light-like momenta $p_k^2 = 0$

⋆ momentum conservation $\sum_k p_k = 0$

⋆ light-like separations $\Delta x_k^2 = 0$

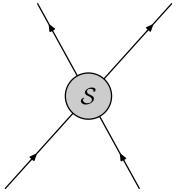
⋆ closure $\sum_k \Delta x_k = 0$

Connections to AdS/CFT Integrability

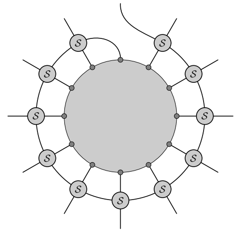
Cusp Dimension from Bethe Equations

Cusp dimension known from AdS/CFT planar integrable system!

Compute cusp dimension using Bethe equations. Integral eq.: [Eden Staudacher]



$$\psi(x) = K(x,0) - \int_0^\infty K(x,y)\,\frac{dy\,y}{\exp(2\pi\lambda^{-1/2}y)-1}\,\psi(y).$$



Kernel $K = K_0 + K_1 + K_\mathrm{d}$ with [NB, Eden Staudacher]

$$K_0(x,y) = \frac{x\,\mathrm{J}_1(x)\,\mathrm{J}_0(y) - y\,\mathrm{J}_0(x)\,\mathrm{J}_1(y)}{x^2-y^2},$$

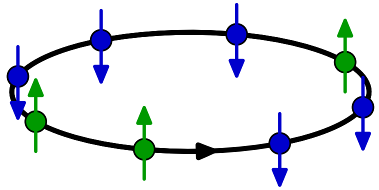
$$K_1(x,y) = \frac{y\,\mathrm{J}_1(x)\,\mathrm{J}_0(y) - x\,\mathrm{J}_0(x)\,\mathrm{J}_1(y)}{x^2-y^2},$$

$$K_\mathrm{d}(x,y) = 2\int_0^\infty K_1(x,z)\,\frac{dz\,z}{\exp(2\pi\lambda^{-1/2}z)-1}\,K_0(z,y).$$

Suitable boundary conditions $\to$ cusp dimension: $D_\mathrm{cusp} = \lambda\pi^{-2}\psi(0)$.

Weak/Strong Expansion

Weak coupling expansion of integral equation



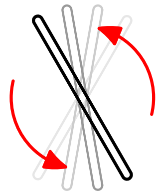
[NB, Eden Staudacher]

$$D_{\text{cusp}}(\lambda) = \frac{1}{2}\frac{\lambda}{\pi^2} - \frac{1}{96}\frac{\lambda^2}{\pi^2} + \frac{11}{23040}\frac{\lambda^3}{\pi^2} - \left(\frac{73}{2580480} + \frac{\zeta(3)^2}{1024\pi^6}\right)\frac{\lambda^4}{\pi^2} \pm \ldots .$$

Agreement with gluon scattering amplitudes. [Bern Dixon Smirnov] [Bern, Czakon, Dixon Kosower, Smirnov]

Strong coupling asymptotic expansion of integral equation [Casteill Kristjansen] [Basso Korchemsky Kotański]

$$E_{\text{cusp}}(\lambda) = \frac{\sqrt{\lambda}}{\pi} - \frac{3\log 2}{\pi} - \frac{\beta(2)}{\pi\sqrt{\lambda}} + \ldots .$$



Agreement with semiclassical energy of spinning string. [Gubser Klebanov Polyakov] [Frolov Tseytlin] [Roiban Tirziu Tseytlin]

Dual Symmetry and Integrability

Same cusp dimension from amplitudes & integrable system:

Question: How to apply integrability to scattering amplitudes?

Strong coupling: Integrability based on $\mathfrak{psu}(2,2|4)$ loop algebra [Bena, Polchinski, Roiban]

$$[\mathfrak{J}^A_m, \mathfrak{J}^B_n] = F^{AB}_C \mathfrak{J}^C_{m+n}.$$

- $\mathfrak{psu}(2,2|4)$ at level 0 ($\mathfrak{J}^A_0$) is ordinary superconformal algebra.
- Different (tilted) embeddings of $\mathfrak{psu}(2,2|4)$ possible.
  Dual superconformal symmetry is such an embedding. [NB, Ricci, Tseytlin, Wolf] [Berkovits, Maldacena]
- Integrability is closure of ordinary and dual superconformal symmetry.

Weak coupling: Integrability based on Yangian symmetry. [Dolan, Nappi, Witten]...

- Yangian action defined for scattering amplitudes at tree level. [Drummond, Henn, Plefka]
- Tree amplitudes are Yangian invariant. [Drummond, Henn, Plefka] [Drummond, Henn, Korchemsky, Sokatchev] [Brandhuber, Heslop, Travaglini]
- Representation consistent with cyclicity.

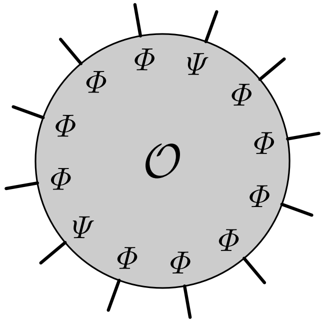
Questions

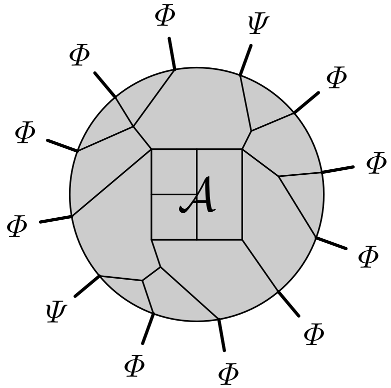
- Can one compute remainder function $F(p, \lambda)$ conveniently?
- Integrability determines $D_{\text{cusp}}(\lambda)$, does it also determine $F(p, \lambda)$?
- Are planar scattering amplitudes uniquely determined from symmetry?
- Can we learn something from integrability for local operators?

Structure of Symmetries

Local Operators vs. Scattering Amplitudes

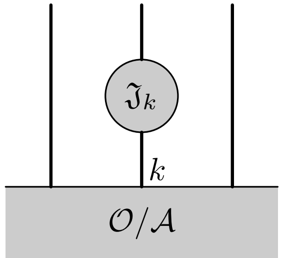
Let us compare single-trace local operators to colour-ordered scattering amplitudes:



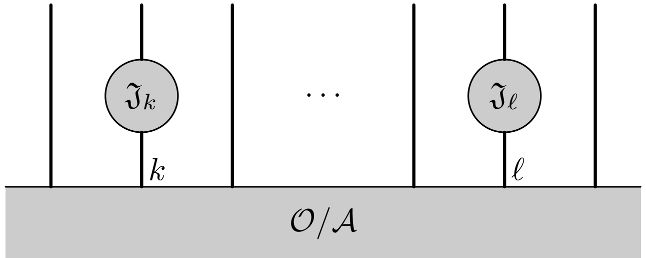


Representations $\mathfrak{J}$ of free superconformal symmetry are analogous:

$$\mathfrak{J} = \sum_{k=1}^{n}$$



$$\widehat{\mathfrak{J}} = \sum_{k<\ell=1}^{n}$$



- Even single-site/leg representation $\mathfrak{J}_k$ is equivalent: field multiplet.
- Integrability: Also action of free Yangian $\widehat{\mathfrak{J}}$ is analogous. [Dolan Nappi Witten] [Drummond Henn Plefka]
- Local operators are multiplets of $\mathfrak{J}$/Amplitudes are invariant under $\mathfrak{J}, \widehat{\mathfrak{J}}$.

Quantum Representation

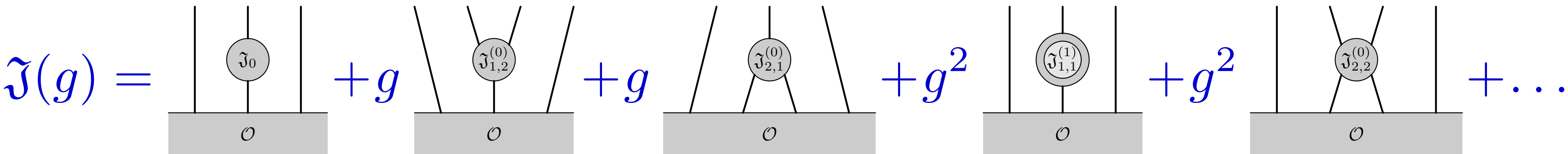
Would like to constrain amplitudes from invariance under $\mathfrak{J}, \widehat{\mathfrak{J}}$.

- Obtain a unique result for planar scattering in $\mathcal{N} = 4$ SYM?

Problem: Amplitudes at loop level are divergent (IR):

- Free superconformal symmetry is broken (scale).
- Unclear how to modify invariance condition in general.

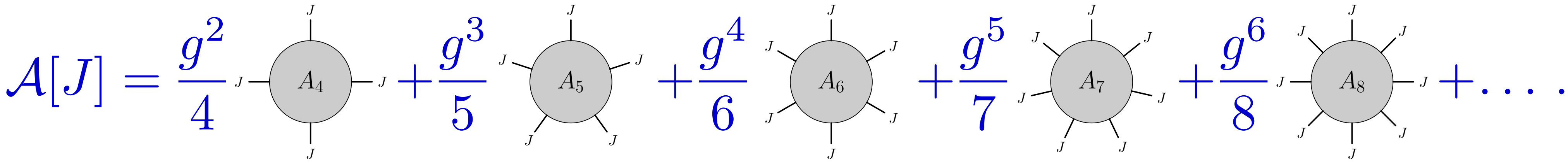
What can we learn from local operators at loop level? $(g \sim \sqrt{\lambda})$



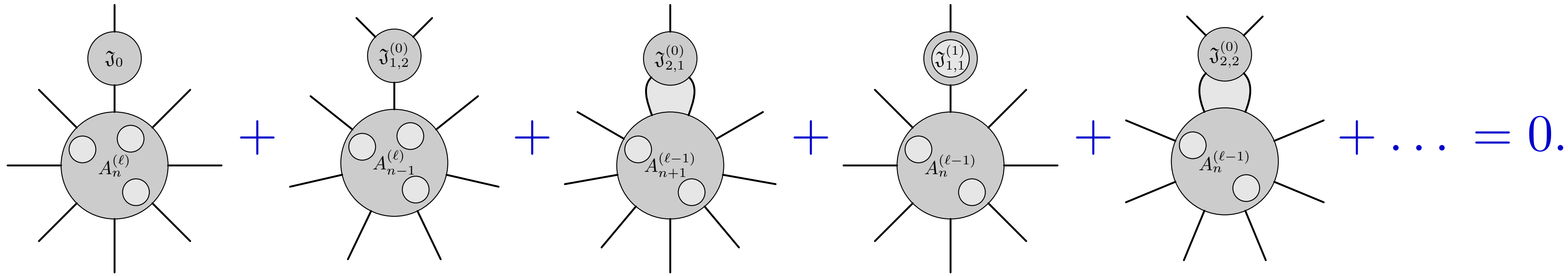
- Free generators deformed by local homogeneous operators.
- Long range action: Acts on multiple sites at the same time. [NB Kristjansen Staudacher]
- Dynamic action: Can change the number of sites. [NB hep-th/0310252]

Amplitude Generating Functional

Package all amplitudes into generating functional with sources $J(p)$



Can now act with dynamic generators. Collect powers of g and legs J

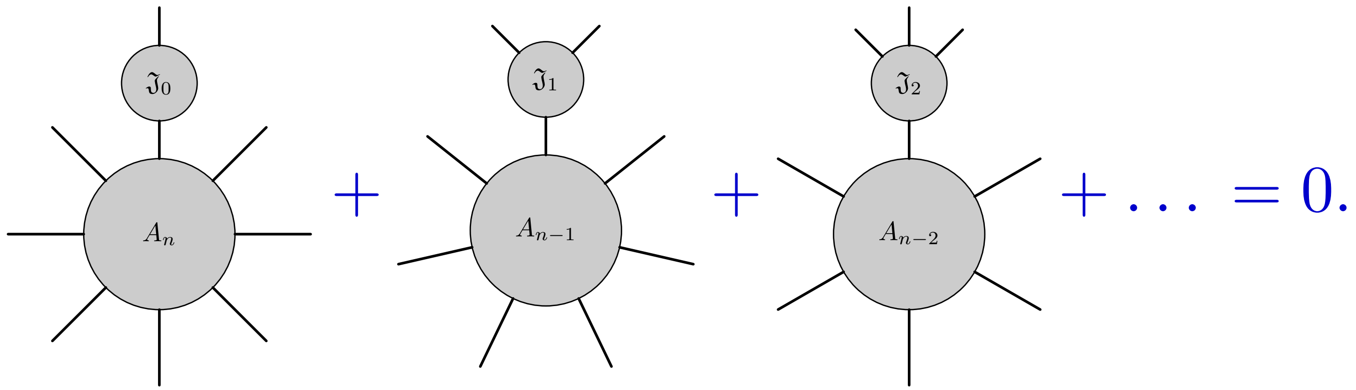


Loops can be formed:

- within the amplitude,
- within the generators and
- between the two.

Tree Level Superconformal Symmetry

Start simple: Restriction to tree level, no loops, various legs



Type of action also known in classical theory as non-linear (in fields).

Puzzle: Trees are invariant under free superconformal action $\mathfrak{J}_0$.
Possible resolutions:

- Trees are also invariant under deformations $(\mathfrak{J}_1 + \mathfrak{J}_2 + \ldots)\mathcal{A} = 0$.
- There are no deformations relevant at tree level $\mathfrak{J}_1 + \mathfrak{J}_2 + \ldots = 0$.
- Trees are actually not invariant under free action $\mathfrak{J}_0\mathcal{A} \neq 0$.

Superconformal Symmetry
for Tree Amplitudes

Spinor Helicity Superspace

Consider on-shell helicity fields $A(p)$, $\psi_a(p)$, $\phi_{ab}(p)$, $\bar\psi^a(p)$, $\bar A(p)$.

Write light-like momentum $p^\mu \sim p^{\dot\alpha\beta}$ using complex bosonic spinors λ^α, $\bar\lambda^{\dot\alpha}$. Combine all fields into superfield using fermionic spinor $\bar\eta^a$

$$p^{\beta\dot\alpha} = \lambda^\beta\bar\lambda^{\dot\alpha}, \quad \Phi(\lambda,\bar\lambda,\bar\eta) = A(\lambda,\bar\lambda) + \bar\eta^a\psi_a(\lambda,\bar\lambda) + \tfrac{1}{2}\bar\eta^a\bar\eta^b\phi_{ab}(\lambda,\bar\lambda) + \ldots$$

Amplitude $A_n(\Lambda_1,\ldots,\Lambda_n)$ becomes function on superspace $\Lambda = (\lambda,\bar\lambda,\bar\eta)$.

Representation of superconformal symmetry on Λ superspace: [Witten hep-th/0312171]

$$\begin{aligned}
\mathfrak{L}^\alpha{}_\beta &= \lambda^\alpha\partial_\beta - \tfrac{1}{2}\delta^\alpha_\beta\lambda^\gamma\partial_\gamma, & \bar{\mathfrak{L}}^{\dot\alpha}{}_{\dot\beta} &= \bar\lambda^{\dot\alpha}\bar\partial_{\dot\beta} - \tfrac{1}{2}\delta^{\dot\alpha}_{\dot\beta}\bar\lambda^{\dot\gamma}\bar\partial_{\dot\gamma},\\
\mathfrak{D} &= \tfrac{1}{2}\partial_\gamma\lambda^\gamma + \tfrac{1}{2}\bar\lambda^{\dot\gamma}\bar\partial_{\dot\gamma}, & \mathfrak{R}^a{}_b &= \bar\eta^a\bar\partial_b - \tfrac{1}{4}\delta^a_b\bar\eta^c\bar\partial_c,\\
\mathfrak{Q}^{\beta a} &= \lambda^\beta\bar\eta^a, & \mathfrak{S}_{\beta a} &= \partial_\beta\bar\partial_a,\\
\bar{\mathfrak{Q}}^{\dot\alpha}_b &= \bar\lambda^{\dot\alpha}\bar\partial_b, & \bar{\mathfrak{S}}^b_{\dot\alpha} &= \bar\eta^b\bar\partial_{\dot\alpha},\\
\mathfrak{P}^{\beta\dot\alpha} &= \lambda^\beta\bar\lambda^{\dot\alpha}, & \mathfrak{K}_{\beta\dot\alpha} &= \partial_\beta\bar\partial_{\dot\alpha}.
\end{aligned}$$

Action on MHV Amplitude

Consider the tree level MHV amplitude in superspace [Parke Taylor] [Berends Giele] [Nair Phys. Lett. B214, 215]

$$A_n^{\mathrm{MHV}} = \frac{\delta^4(P)\,\delta^8(Q)}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}, \qquad P^{\beta\dot\alpha} = \sum_{k=1}^{n} \lambda_k^{\beta}\bar\lambda_k^{\dot\alpha}, \quad Q^{\beta a} = \sum_{k=1}^{n} \lambda_k^{\beta}\bar\eta_k^{a}.$$

The twistor brackets are defined as $\langle\lambda,\mu\rangle = \varepsilon_{\alpha\beta}\lambda^\alpha\mu^\beta$, $[\bar\lambda,\bar\mu] = \varepsilon_{\dot a\dot b}\bar\lambda^{\dot\alpha}\bar\mu^{\dot\beta}$.

Apply the free superconformal generator $(\bar{\mathfrak{S}}_0)^b_{\dot\alpha} = \sum_{k=1}^n \bar\eta_k^b \bar\partial_{k,\dot\alpha}$.
The anti-holomorphic derivative $\bar\partial_{\dot\alpha} = \partial/\partial\bar\lambda^{\dot\alpha}$ acts only on $\bar\lambda^{\dot\beta}$ in $\delta^4(P)$

$$(\bar{\mathfrak{S}}_0)^b_{\dot\alpha}\delta^4(P) = \sum_{k=1}^{n} \eta_k^b\bar\partial_{k,\dot\alpha}\delta^4(P) = \sum_{k=1}^{n} \eta_k^b\lambda_k^\gamma \frac{\partial\delta^4(P)}{\partial P^{\gamma\dot\alpha}} = Q^{\gamma b}\frac{\partial\delta^4(P)}{\partial P^{\gamma\dot\alpha}}.$$

This term vanishes due to $\delta^8(Q)$. Thus $\bar{\mathfrak{S}}_0 A^{\mathrm{MHV}} = 0$.

Holomorphic Anomaly

Not quite! The holomorphic anomaly yields extra contributions [Witten hep-th/0312171]

$$\frac{\partial}{\partial\bar z}\frac{1}{z} = \pi\delta^2(z), \qquad \frac{\partial}{\partial\bar\lambda^{\dot\alpha}}\frac{1}{\langle\lambda,\mu\rangle} = \pi\delta^2(\langle\lambda,\mu\rangle)\,\varepsilon_{\dot\alpha\dot\beta}\bar\mu^{\dot\beta}.$$

Taking the anomaly into account one obtains [Bargheer, NB, Galleas Loebbert, McLoughlin]

$$(\bar{\mathfrak{S}}_0)^b_{\dot\alpha}A_n^{\text{MHV}} = -\pi\sum_{k=1}^{n}\varepsilon_{\dot\alpha\dot\gamma}\big(\bar\lambda_k^{\dot\gamma}\bar\eta_{k+1}^b - \bar\lambda_{k+1}^{\dot\gamma}\bar\eta_k^b\big)\frac{\delta^2(\langle k,k+1\rangle)\,\delta^4(P)\,\delta^8(Q)}{\langle 12\rangle\ldots\langle k,k+1\rangle^0\ldots\langle n1\rangle}.$$

Free conformal symmetry breaks down when $\lambda_k \sim \lambda_{k+1}$: Collinear!
Note that gap $\langle k,k+1\rangle^0$ in denominator chain of $\langle j,j+1\rangle$ closes

$$\langle\lambda_{k-1},\lambda_k\rangle\langle\lambda_{k+1},\lambda_{k+2}\rangle \sim \langle\lambda_{k-1},\lambda_{k,k+1}\rangle\langle\lambda_{k,k+1},\lambda_{k+2}\rangle.$$

Remainder contains $A_{n-1}^{\text{MHV}}(\Lambda_1,\ldots,\Lambda_{k,k+1},\ldots,\Lambda_n)$. Calls for $\mathfrak{J}_1$!

Symmetry Correction

Formulate by acting on generating functional $(\bar{\mathfrak{S}}_0)^b_{\dot\alpha}\mathcal{A}^{\mathrm{MHV}}_n[J]$

$$\ldots = -\pi \int d^{4|4}\Lambda_{12} \prod_{k=3}^{n} d^{4|4}\Lambda_k\, d^4\bar\eta'\, d\alpha\, d\varphi\; e^{3i\varphi}\varepsilon_{\dot\alpha\dot\gamma}\bar\lambda^{\dot\gamma}_1\eta^b_2$$
$$\times \operatorname{Tr}\big([J(\Lambda_1), J(\Lambda_2)]\ldots J(\Lambda_n)\big)\; A^{\mathrm{MHV}}_{n-1}(\Lambda_{12}, \Lambda_3, \ldots, \Lambda_n)$$

with the replacements

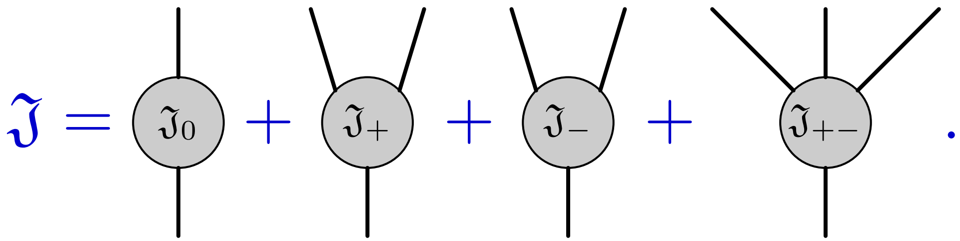
$$\lambda_1 = \lambda_{12}e^{i\varphi}\sin\alpha, \qquad \bar\eta_1 = (\bar\eta_{12}\sin\alpha + \bar\eta'\cos\alpha)e^{-i\varphi},$$
$$\lambda_2 = \lambda_{12}\cos\alpha, \qquad \bar\eta_2 = \bar\eta_{12}\cos\alpha - \bar\eta'\sin\alpha.$$

Compensate anomaly by $(\bar{\mathfrak{S}}_+)^b_{\dot\alpha}\mathcal{A}^{\mathrm{MHV}}_{n-1}[J]$ with the action on sources

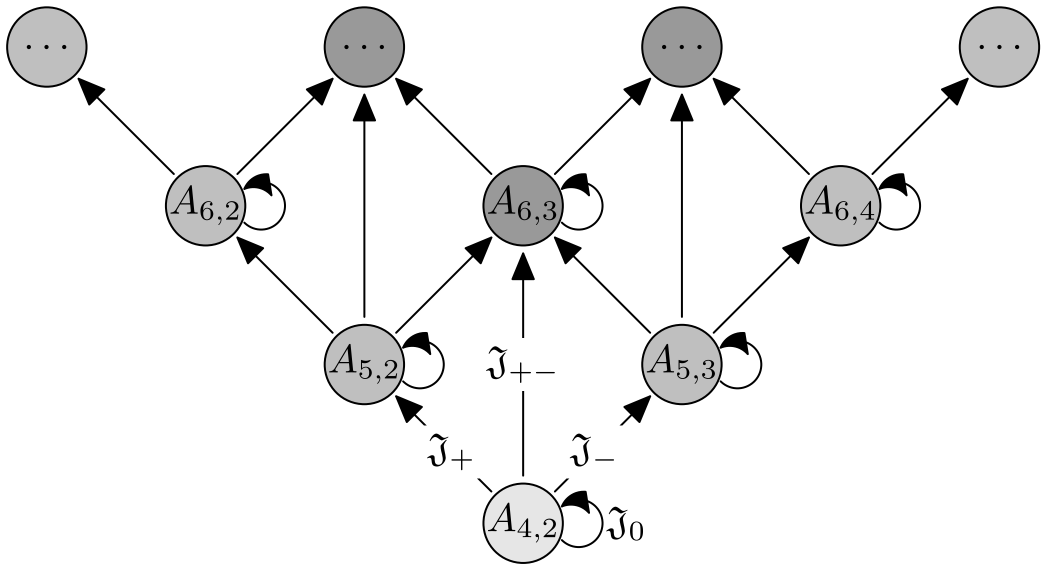
$$(\bar{\mathfrak{S}}_+)^b_{\dot\alpha}J(\Lambda_{12}) = \pi \int d^4\bar\eta'\, d\alpha\, d\varphi\; e^{3i\varphi}\varepsilon_{\dot\alpha\dot\gamma}\bar\lambda^{\dot\gamma}_1\bar\eta^b_2\,[J(\Lambda_1), J(\Lambda_2)].$$

Classical Representation

We find the following corrections for $\mathfrak{S}$ (only $-$), $\bar{\mathfrak{S}}$ (only $+$) and $\mathfrak{K}$ (all)



Collinear anomalies removed from all tree amplitudes: [Bargheer, NB, Galleas Loebbert, McLoughlin]



Multi-particle singularities carry no anomalies.

Symmetry Algebra

Does the deformed algebra close? [Bargheer, NB, Galleas, Loebbert, McLoughlin]

- $[\mathfrak{L}, \ldots]$, $[\bar{\mathfrak{L}}, \ldots]$: proper index contractions.
- $[\mathfrak{D}, \ldots]$: proper conformal weights.
- $\{\mathfrak{Q}, \bar{\mathfrak{S}}\}$, $\{\bar{\mathfrak{Q}}, \mathfrak{S}\}$: almost trivial.
- $\{\mathfrak{Q}, \mathfrak{S}\}$, $\{\bar{\mathfrak{Q}}, \bar{\mathfrak{S}}\}$: integrand asymmetric in α.
- $\{\mathfrak{S}, \mathfrak{S}\}$, $\{\bar{\mathfrak{S}}, \bar{\mathfrak{S}}\}$: rather non-trivial, requires vanishing central charge. Closes only onto gauge transformation

$$\mathfrak{G}_{ab} J(\Lambda) \sim [\partial_a \partial_b J(0), J(\Lambda)].$$

- $\{\mathfrak{S}, \bar{\mathfrak{S}}\}$: defines $\mathfrak{K}$.
- $[\mathfrak{K}, \ldots]$, $[\mathfrak{P}, \ldots]$: through Jacobi identities.

Superconformal algebra satisfied, but contains gauge transformations.

Conclusions

Conclusions

⋆ **Interacting Superconformal Symmetry**

- Tree amplitudes almost invariant under free superconformal symmetry.
- Invariance violated for singular configurations: Collinear momenta.
- Transformations can be corrected to make trees fully invariant.
- Dynamic corrections requires, changes number of legs.
- Superconformal algebra closes onto gauge transformations.

⋆ **Open Problems**

- Show that tree amplitudes are uniquely determined by Yangian.
- Promote to loop level.
- What about conformal inversions?
- Apply to $\mathcal{N} < 4$ supersymmetric gauge theories with matter?
- Similar effects for $\mathrm{E}_{7(7)}$ in supergravity?