On the bound states in QFT.

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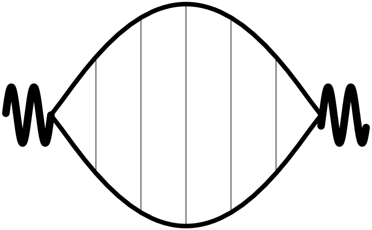
- Bound states in QFT is not well formulated problem up to now.

- A bound state is a simple pole of an elastic scattering amplitude in a canal with appropriate quanum numbers.
    1. Quantum electrodynamics of electrons
    2. The Bethe-Salpeter (BS) equation in the ladder approximation is a direct way to study this problem.
    3. BS equation is not gauge invariant. Choice of gauge.
    4. Positronium mass in Feynman and Coulomb gauges.

- A bound state is a simple pole of a Green function of currents with appropriate quanum numbers.
    1. Quantum electrodynamics of charged scalar particles.
    2. Functional integral representation
    3. Relativistic corrections to the non-relativistic Schrödinger equation.
    4. Bound state mass for large α.

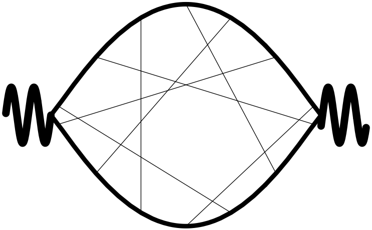
Bethe-Salpeter equation

Gauge non-invariant approach



Functional integral representation

Gauge invariant approach



Bethe-Salpeter equation
QUANTUM ELECTRODYNAMICS

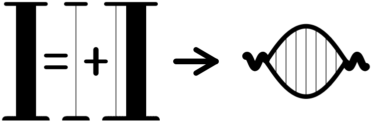
$$L = -\frac{1}{4} F^2_{\mu\nu}(x) + (\overline{\psi}(x)(\hat{p} + e\hat{A}(x) - m)\psi(x)),$$

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x).$$

Feynman gauge : $\tilde{D}_{\mu\nu}(k) = \frac{\delta_{\mu\nu}}{k^2}$

Coulomb gauge : $\tilde{D}_{\mu\nu}(k) = \begin{cases} \left[\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right] \frac{1}{k^2} \\ \\ -\frac{1}{\mathbf{k}^2} \end{cases}$

Bethe-Salpeter equation



- $\star$ The BS kernel in a symmetric form looks as $K = K_0 + K_I$
  - $\mathrm{Tr}\ K_0^2 = \infty$ is of the "fall at center"potential type $\Rightarrow$ $\frac{\alpha}{\pi} < \frac{\alpha_c}{\pi} \sim 1$
  - $\mathrm{Tr}\ K_I^2 < \infty$ is responsible for bound states

- $\star$ Variational procedure of calculations can be used.

- $\star$ The binding energy of the 1^--state (positronium) is calculated.

Binding energy ϵ (eV) of the 1^- state

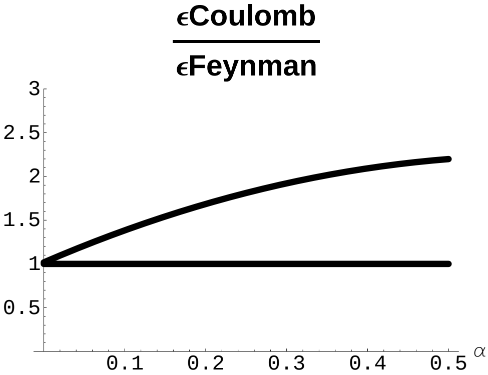
α	0.0005	0.001	$\frac{1}{137}$ 0.0073	0.01	0.1	0.3	0.5
Feynman	0.032	0.126	6.47	12.0	893	5 700	12 600
Coulomb	0.032	0.127	6.8	12.8	1 270	10 800	27 800
Schrödinger $\frac{\alpha^2}{4} m_e$	0.032	0.127	6.8	12.8	1 280	11 500	31 900

$$Feynman \qquad V \qquad \rightarrow \mathcal{J}_j(x) = \left(\overline{\Psi}(x) V(\overleftrightarrow{p}_x) \gamma_j \Psi(x)\right)$$

$$Coulomb \qquad V + iT \qquad \rightarrow \mathcal{J}_j(x) = \left(\overline{\Psi}(x) V(\overleftrightarrow{p}_x)(1 + \gamma_0) \gamma_j \Psi(x)\right)$$

- The gauge invariance is broken in the Bethe-Salpeter equation with any fixed kernel

- The Feynman and Coulomb gauges give different results



- Problem $\Longrightarrow$ What gauge should be chosen?

- May be there exists a preferable gauge in the bound state problem?

- Standard choice $\Longrightarrow$ Coulomb gauge.

Functional integral approach
QUANTUM SCALAR ELECTRODYNAMICS

$$S[\Phi^+, \Phi, \phi] = \int dx \left[-\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \Phi^+ [(i\partial_\mu + eA_\mu)^2 + m^2] \Phi \right]$$

The polarization operator

$$\Pi(x-y) = \iint D\Phi D\Phi^+ D\phi \cdot \Phi^+(x)\Phi(x)\Phi^+(y)\Phi(y) \cdot e^{S[\Phi^+, \Phi, \phi]}$$

$$= \int DA \delta[\partial_\mu A_\mu] e^{-\frac{1}{4}\int dx \; F_{\mu\nu} F_{\mu\nu}} \cdot S(x, y|A) S(y, x|A) \sim e^{-M|x-y|}$$

$$S(x, y|A) = \frac{1}{(i\partial_\mu + eA_\mu(x))^2 + m^2} \cdot \delta(x-y)$$

$$= \int_0^\infty \frac{ds}{8\pi^2 s^2} e^{-\frac{1}{2}\left[m^2 s + \frac{(x-y)^2}{s}\right]} \cdot \int D\xi e^{-\int_0^s d\tau \frac{\dot{\xi}^2(\tau)}{2} + ie\int_0^s d\tau \dot{z}_\mu(\tau) A_\mu(z(\tau))}$$

$$z(\tau) = x\frac{\tau}{s} + y\left(1 - \frac{\tau}{s}\right) + \xi(\tau)$$

$$M = 2m\sqrt{1-\varepsilon(\alpha)}$$

$$J = e^{x\varepsilon(\alpha)} = \int \frac{D\xi}{C} e^{-\frac{1}{2}\int_0^{x\alpha^2} d\tau\left[\dot{\boldsymbol{\xi}}_1^2(\tau)+\dot{\boldsymbol{\xi}}_2^2(\tau)+\dot{\xi}_1^2(\tau)+\dot{\xi}_2^2(\tau)\right]+\mathcal{W}[\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\xi_1,\xi_2]},$$

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$$W[\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \xi_1, \xi_2]$$

$$= \iint_0^{x\alpha^2} d\tau_1 d\tau_2 \left[\left(1+\alpha\dot{\xi}_1(\tau_1)\right)\left(1+\alpha\dot{\xi}_2(\tau_2)\right) + \alpha^2 \dot{\boldsymbol{\xi}}_1(\tau_1)\dot{\boldsymbol{\xi}}_2(\tau_2)\right]$$

$$\cdot \int \frac{d\mathbf{q}}{2\pi^2} \int \frac{dq\ e^{iq(\tau_1-\tau_2)}}{2\pi} \frac{e^{iq(\alpha(\xi_1(\tau_1)-\xi_2(\tau_2)))+i\mathbf{q}(\boldsymbol{\xi}_1(\tau_1)-\boldsymbol{\xi}_2(\tau_2))}}{\mathbf{q}^2+\alpha^2 q^2}$$

$$\Longrightarrow \int_0^{x\alpha^2} d\tau \frac{1+\alpha^2\dot{\boldsymbol{\xi}}_1(\tau)\dot{\boldsymbol{\xi}}_2(\tau)}{|\boldsymbol{\xi}_1(\tau)-\boldsymbol{\xi}_2(\tau)|}$$

Potential and non-potential corrections.

Gaussian equivalent Representation

$$\int \frac{D\phi}{\sqrt{\det D}} e^{-\frac{1}{2}(\phi D^{-1}\phi)+W[\phi]} \equiv e^{W_0} \int \frac{D\phi}{\sqrt{\det S}} e^{-\frac{1}{2}(\phi S^{-1}\phi)+W_I[\phi]},$$

$$W[\phi] = \int d\mu_b e^{i(b\phi)} = \iint_0^x d\tau_1 d\tau_2 D(n(\tau_1 - \tau_2) + (\phi(\tau_1) - \phi(\tau_2)))$$

$$\star\ \mathbf{W}_I[\phi] = \int d\mu_b e^{-\frac{1}{2}(bSb)} \vdots \left[e^{i(b\phi)} - 1 + \frac{1}{2}(b\phi)^2 \right] \vdots_S = O(\phi^4),$$

$$\star\ W_0 = \frac{1}{2} \ln \frac{\det S}{\det D} - \frac{1}{2}([D^{-1} - S^{-1}]S) + \int d\mu_b e^{-\frac{1}{2}(bSb)}$$

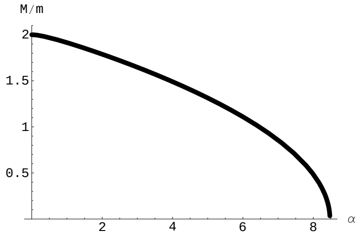
$$\star\ D^{-1}(x_1, x_2) - S^{-1}(x_1, x_2) + \int d\mu_b b(x_1) b(x_2) e^{-\frac{1}{2}(bSb)} = 0$$

Mass of the relativistic bound state

$$M = 2m\sqrt{1-\varepsilon(\alpha)}$$

$$\varepsilon(\alpha) = \begin{cases} \frac{\alpha^2}{4} & \alpha \ll 1 \\ \alpha \cdot \text{const} & \alpha \gg 1 \end{cases}$$

$$\varepsilon(\alpha) \sim \frac{1}{4}\frac{\alpha^2}{1+2\alpha}$$



RESULTS

1. For small coupling constant $\alpha \ll 1$ all approaches: non-relativistic Schrödinger equation, Bethe-Salpeter equation in the ladder approximation and functional representation give the same result.

2. Relativistic corrections, i.e. next orders α corrections, to the non-relativistic Schrödinger equation have no potential character, they contain time dependent terms.

3. Gauge invariance is broken in the Bethe-Salpeter equation in the ladder approximation. Problem is what gauge should be chosen. Standard choice - Coulomb gauge.

4. Functional approach $\Longrightarrow$ gauge invariant approach.
Problem is how to calculate functional integrals.
Gaussian equivalent representation $\Longrightarrow$ weak and strong coupling regime calculations.