Self-consistent current structures in relativistic collisionless plasmas:

Exact solutions for broad classes of particle distributions

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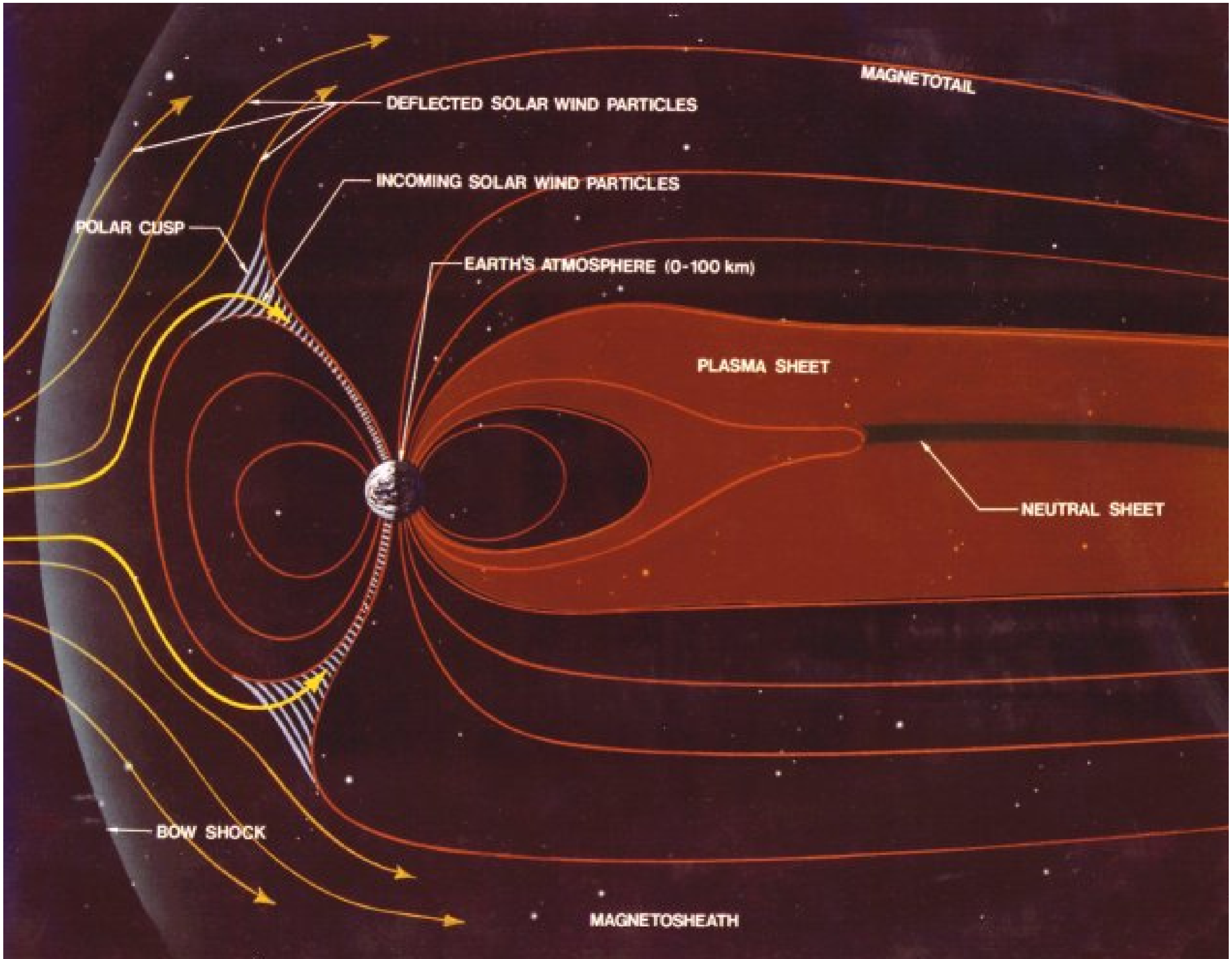
Recent progress in analytic understanding of origin and various properties of self-consistent quasi-static configurations of magnetic field and current structures emerging in anisotropic collisionless plasma is reviewed and applied to the problem of relativistic shocks in astrophysical plasmas.

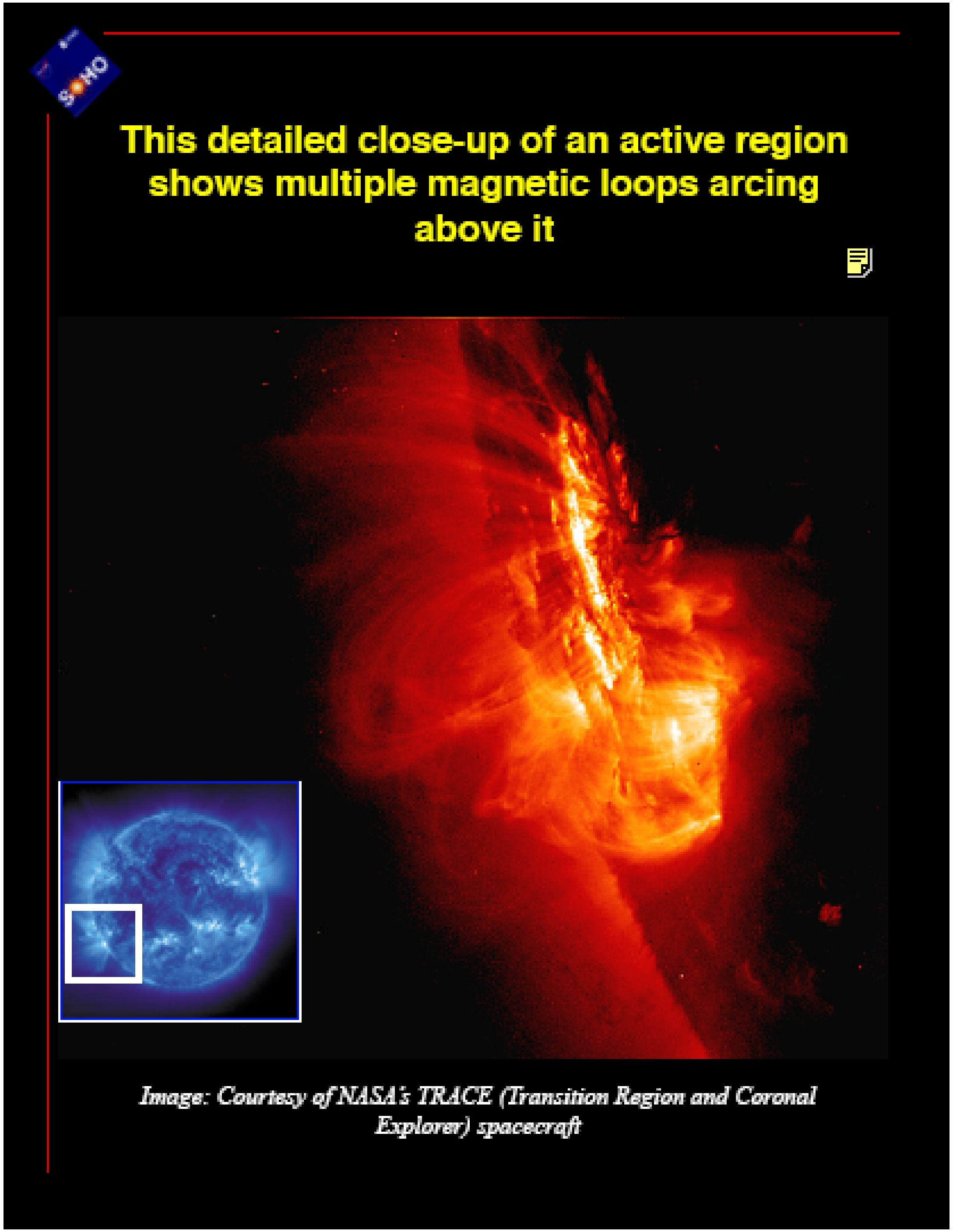
In typical planar and cylindrical geometries, we find analytically a wide class of nonlinear stationary current structures which can be equally easy realized in relativistic and non-relativistic collisionless plasma. These solutions are based on the method of integrals of motion, and extend far beyond the known generalizations of non-relativistic Harris model. The obtained Grad-Shafranov type equations allow to analytically investigate and compare general properties and possible evolution of these structures. Among the properties of newly found stationary solutions we discuss the ratio of magnetic field energy to that of particles, the anisotropy of particle momentum distribution, the spatial scales and profiles of particle density, current and magnetic field, etc.

We carry out also the short wavelength instability analysis of these current sheets and filaments in the regions with small enough magnetic fields where this instability is expected to be the most pronounced. We point to peculiarities of the synchrotron radiation spectra of relativistic particles in self-consistent current sheets, which make it possible to study the structure and evolution of their currents and magnetic fields based on the observed radiation.

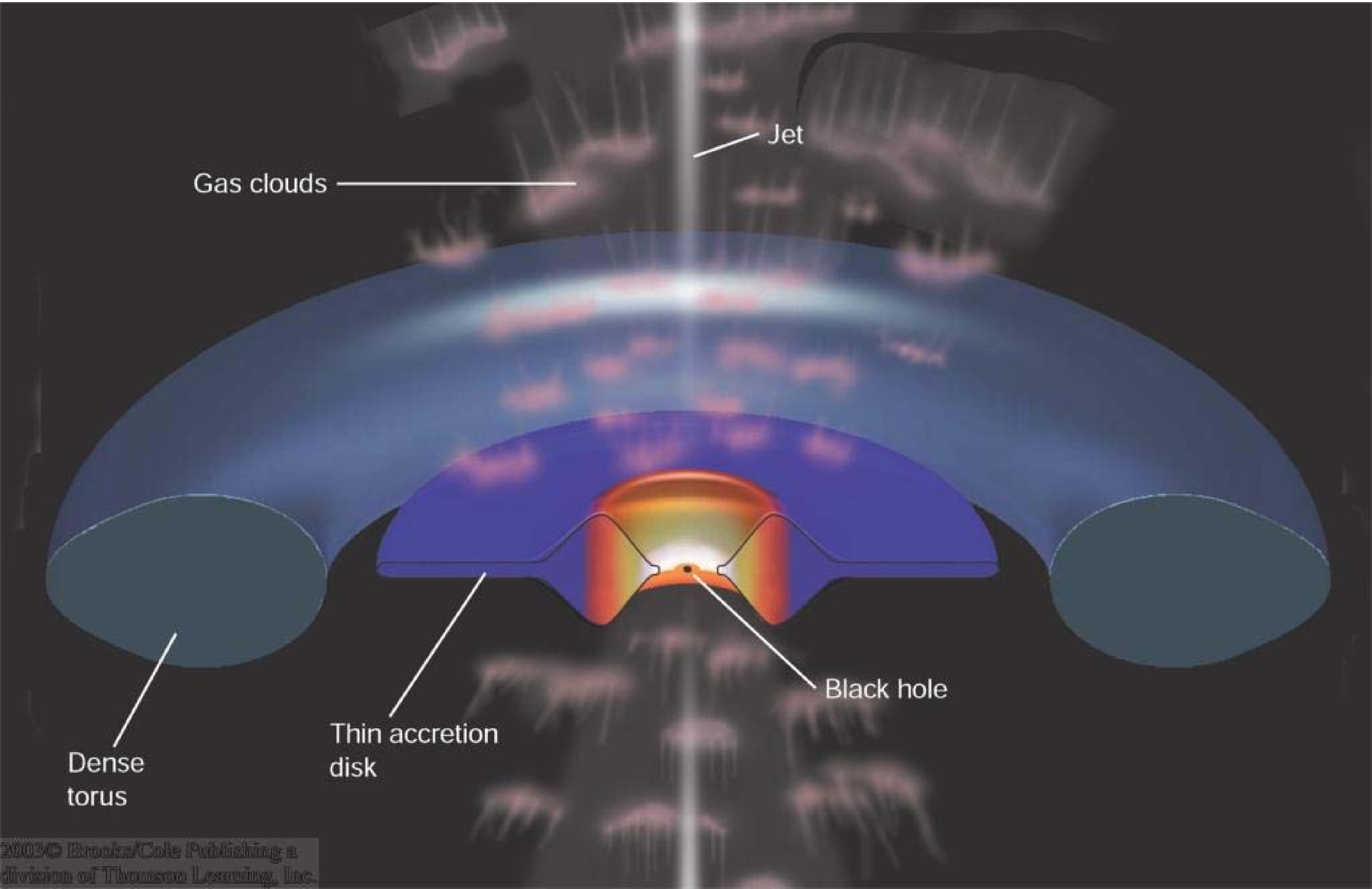
Current sheet in Earth's magnetosphere



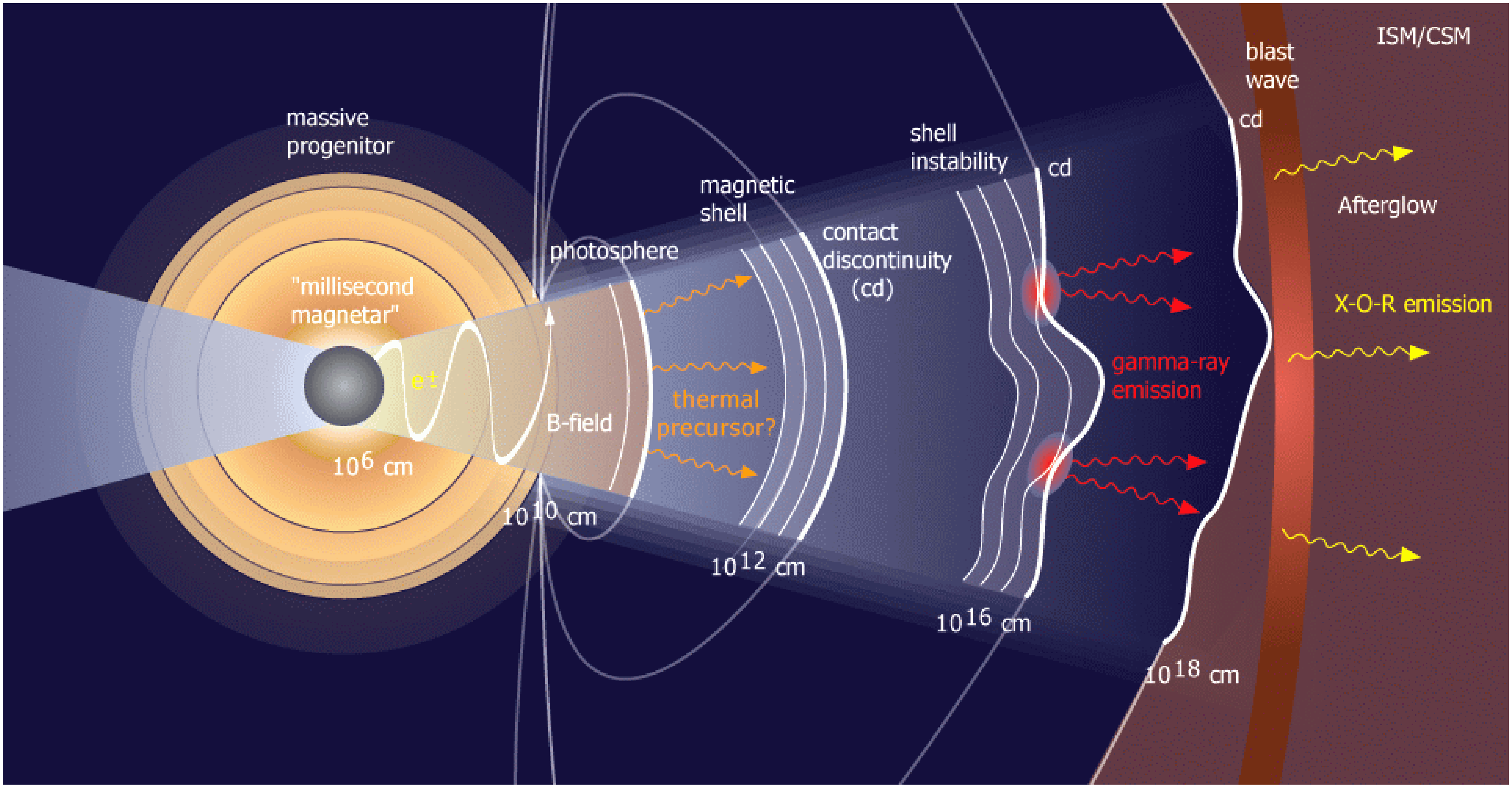
Current sheets and filaments in solar corona



Possible sites of relativistic collisionless plasma



Relativistic shock model of Gamma-Ray Bursts



Typical parameters (relativistic plasma)

- Active galactic nuclei (blazars)

  $\Gamma\sim5\text{-}20$, $B\sim0.1G$, $N\sim10cm^{-3}$, $\gamma\sim10^5$, $L\sim10^{16}cm$

- Microquasars (relativistic)

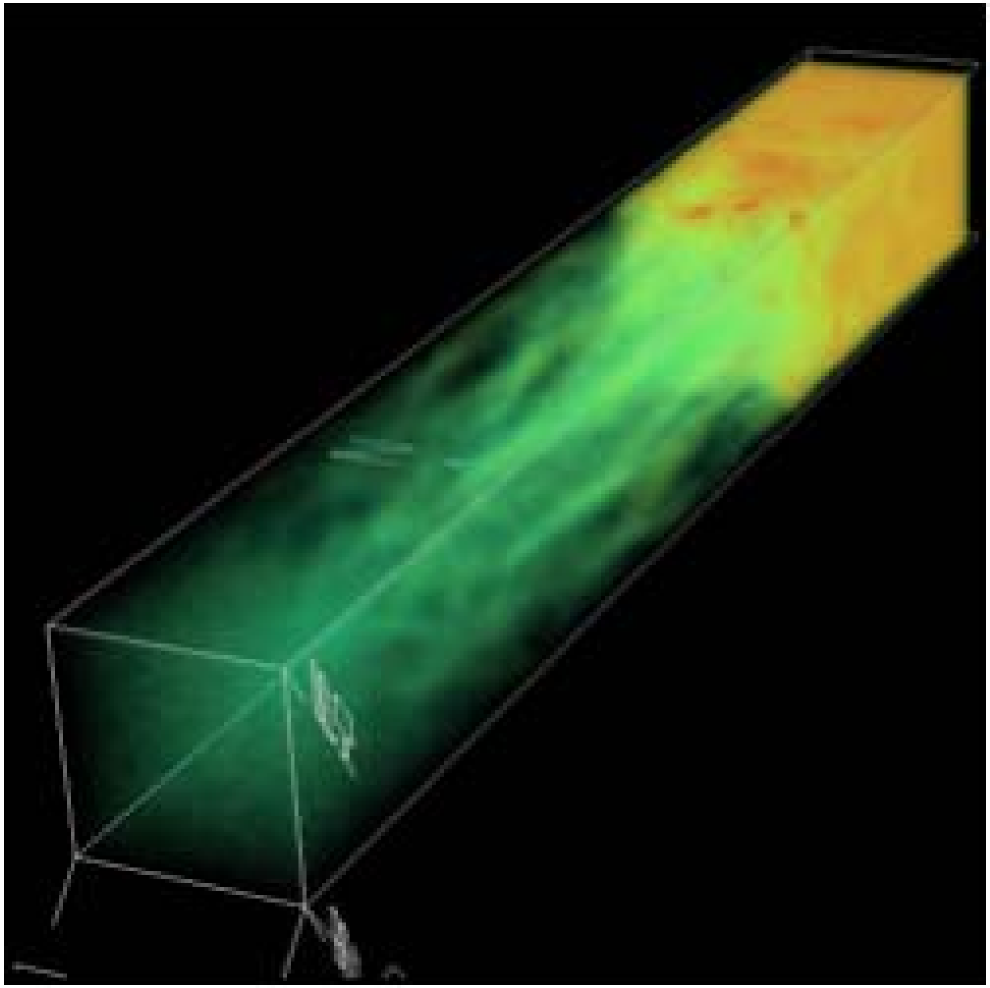
  $\Gamma\sim3$, $B\sim10^6G$, $N\sim10^{15}cm^{-3}$, $\gamma\sim10^2$, $L\sim10^9cm$

- Gamma-ray bursts (long)

  $\Gamma\sim300$, $B\sim10^6G$, $N\sim10^{15}cm^{-3}$, $\gamma\sim10^3$, $L\sim10^{14}cm$

Current structures in relativistic collisionless shocks

(Numerical simulation, A.Spitkovsky, 2006)



Numerical simulations of magnetic structure formation

Particle-in-cell experiments in 2D and 3D

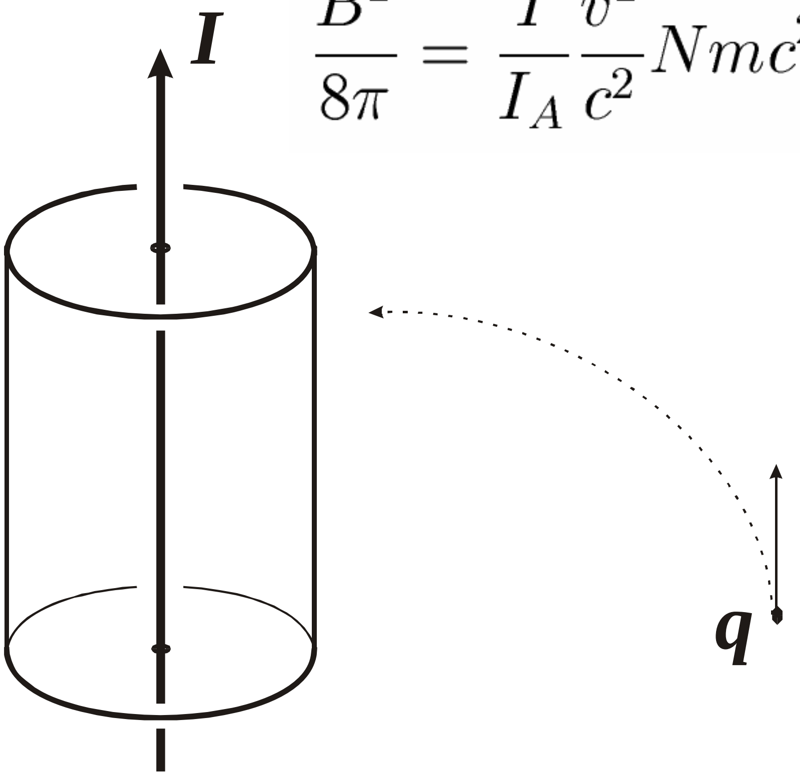
- A. Pukhov, Rep. Prog. Phys. 2003, **66**, 47.
- L. Silva *et al*, ApJ 2003, **596**, L121.
- F. Califano, D.D. Sarto, F. Pegoraro, PRL **96**, 105008 (2006).
- K.-I. Nishikawa, C.B. Hededal *et al*., ApJ **642**, n. 2, 1267 (2006).
- T.N. Kato, Phys. Plasmas **12**, 080705 (2005).
- A. Spitkovsky, ApJ **673**, 1, L39 (2008); U. Keshet *et al*., ApJ **693**, L127 (2009); A. Spitkovsky, L. Sironi, arXiv:0901.2578 (2009).

Alfven current and filament self-limitation

$$I_A = \frac{mc^3}{e}\frac{v}{c}\gamma$$

$$\frac{r}{r_H} = \frac{2I}{I_A}$$

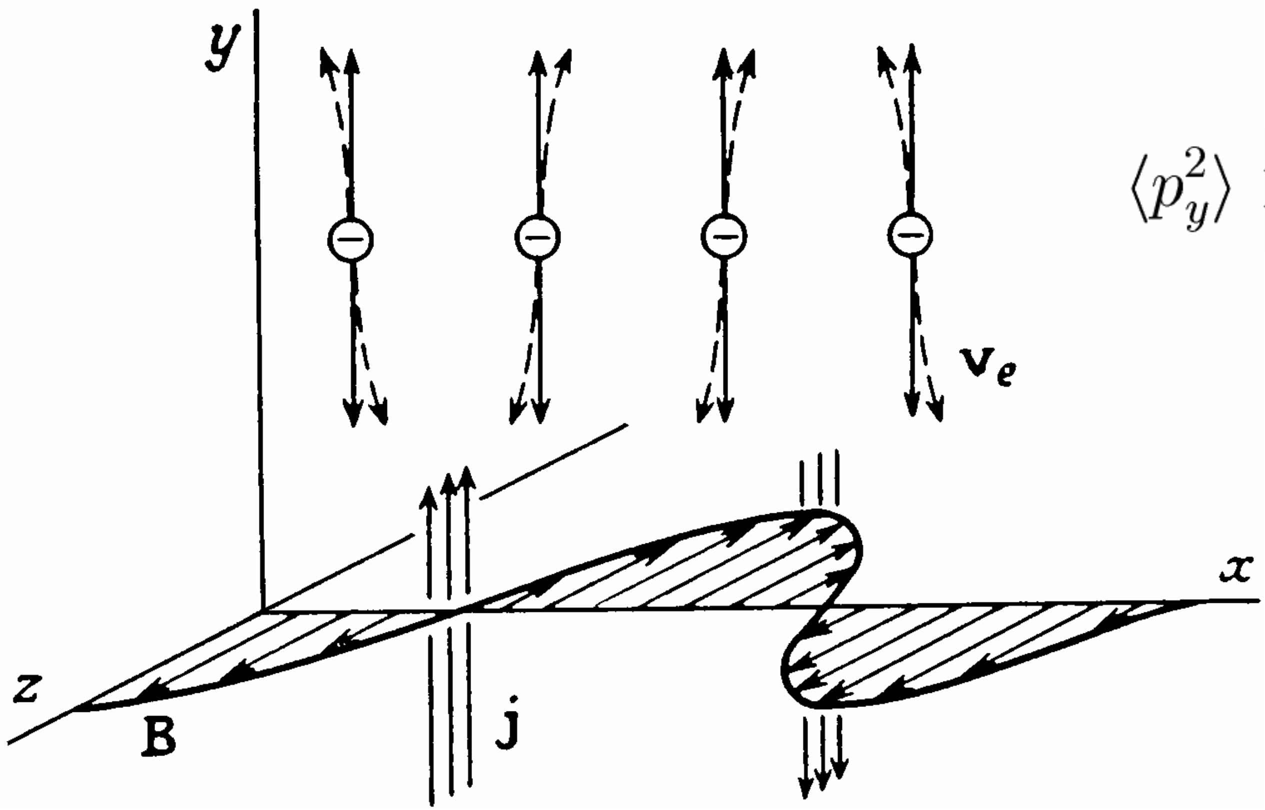
$$I \sim I_A \cdot \frac{D}{r_{H\min}}$$



$$\frac{B^2}{8\pi} = \frac{I}{I_A}\frac{v^2}{c^2}Nmc^2\gamma$$

$$\frac{B^2}{8\pi} < Nmc^2\gamma$$

Weibel instability



$$\langle p_y^2 \rangle > \langle p_x^2 \rangle$$

Highly anisotropic velocity distributions are unstable

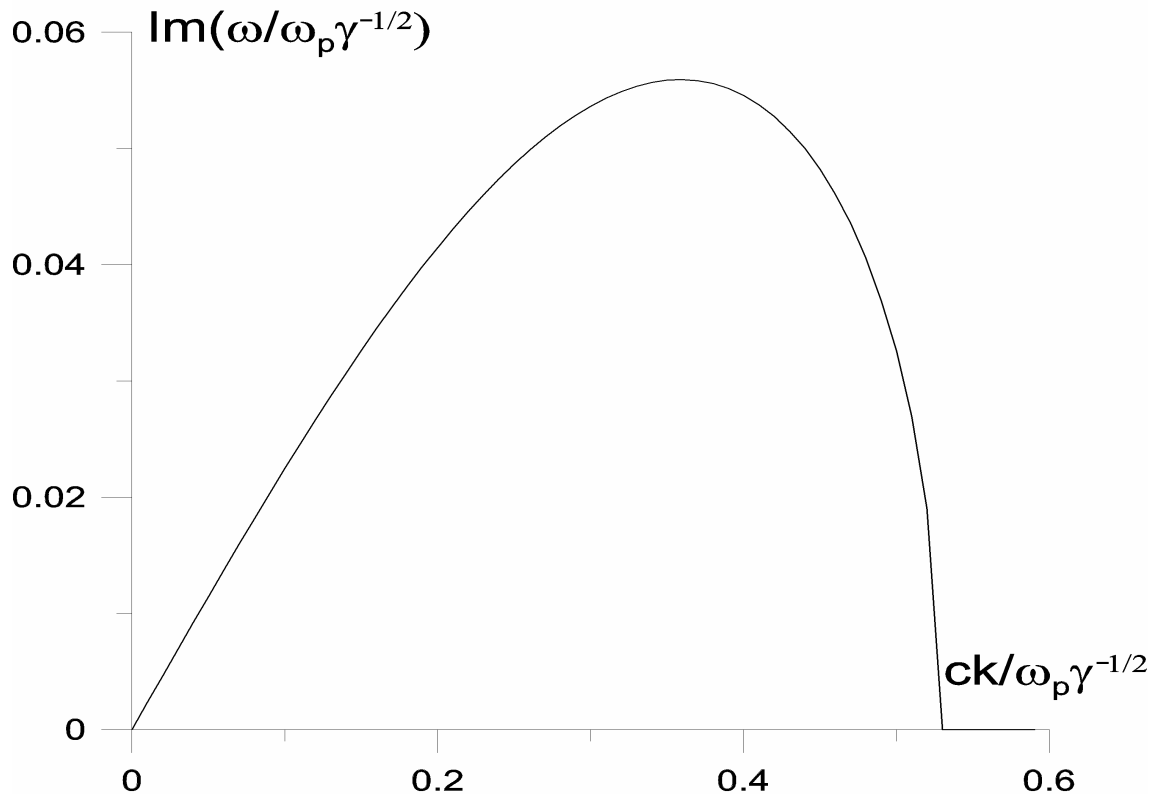
$$\max(\mathrm{Im}\,\omega) \sim \omega_p / \sqrt{\gamma}$$

Dispersion relation

$$1 - \frac{c^2k^2}{\omega^2} + \sum_\alpha \frac{4\pi e_\alpha^2}{m_\alpha \omega^2} \int \left[\frac{p_y k_x}{\gamma_a m_\alpha} \frac{\partial f_{0\alpha}}{\partial p_x} + \right.$$

$$\left. + \left(\omega - \frac{\mathbf{kp}}{\gamma_a m_\alpha} \right) \frac{\partial f_{0\alpha}}{\partial p_y} \right] \frac{p_y}{(\omega - \mathbf{kp}/\gamma_a m_\alpha)} \frac{d^3\mathbf{p}}{\gamma_a} = 0$$

Typical growth rate graph



- The instability is aperiodic, $\mathrm{Re}\,\omega = 0$
- Clearly defined spatial scale, $c\sqrt{\gamma}/\omega_p << l_{free\ path}$

Instability condition

$$\omega = 0,\ k \neq 0 \Rightarrow$$

$$\sum_\alpha \frac{e_\alpha^2}{m_\alpha} \int \left[1 + \frac{p_y^2 k^2}{(\mathbf{kp})^2} \right] \frac{f_{0\alpha}}{\gamma_a} d^3\mathbf{p} < 0$$

$$\langle p_y^2 \rangle > \langle p_x^2 \rangle$$

Saturation of instability

- PDF isotropization due to particle gyration

$$B_z \lesssim \frac{mc\gamma\,|\omega|}{e}$$

- Particle displacement along wavevector ***k***

$$B_z \lesssim \frac{mc\gamma\,|\omega|}{e}\,\frac{|\omega|}{kv}$$

- Particle deceleration by electric field ***E***

$$B_z \lesssim \frac{mc\gamma\,|\omega|}{e}\,\frac{kv}{|\omega|}$$

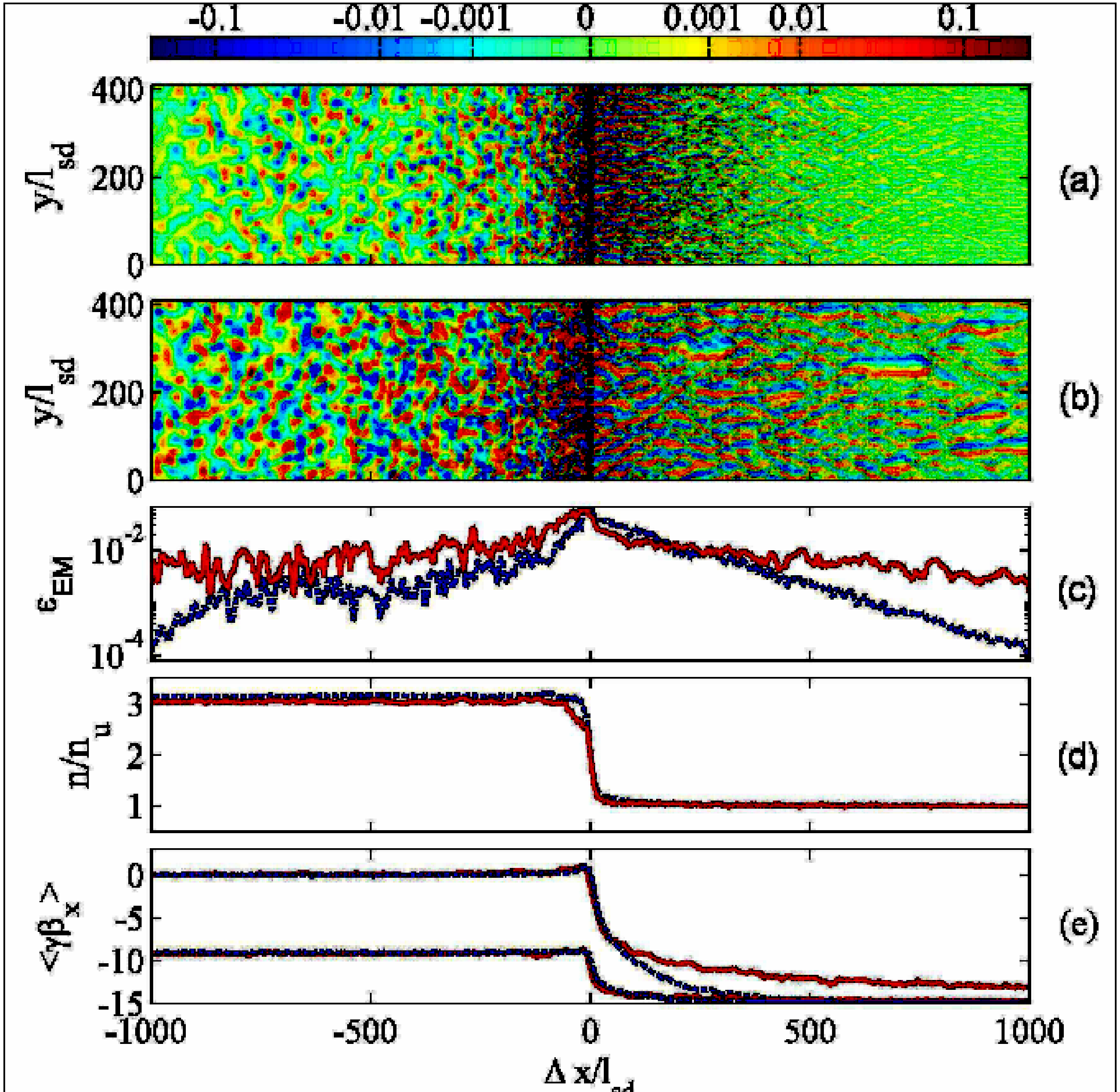
Nonlinear evolution

- Quasineutrality
- Magnetic energy can approach equipartition
- Current filaments merge due to Ampère force
- Spatial scale increases
- Slow magnetic field decay
- Metastable configurations

Equal treatment of relativistic and non-relativistic plasma

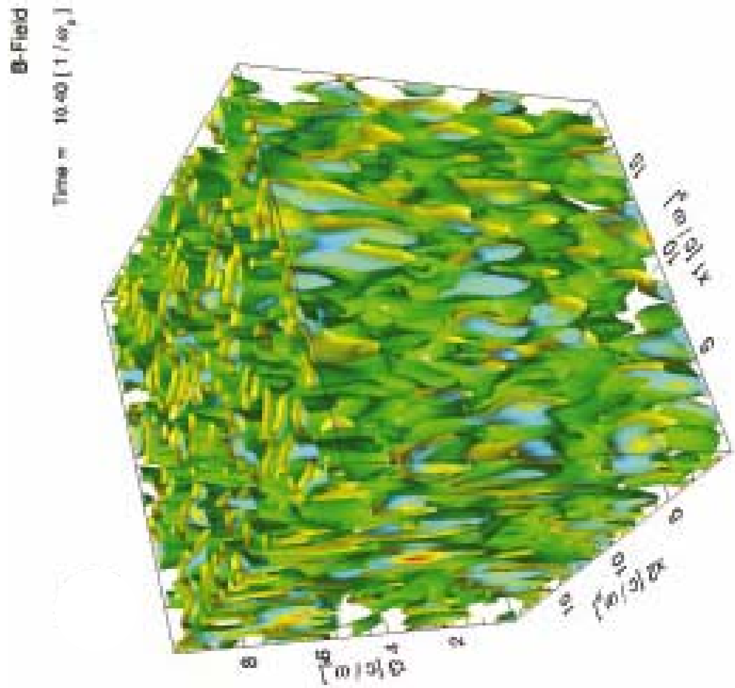
$$\langle B^2/8\pi \rangle \lesssim \langle (\gamma - 1) N m c^2 \rangle$$

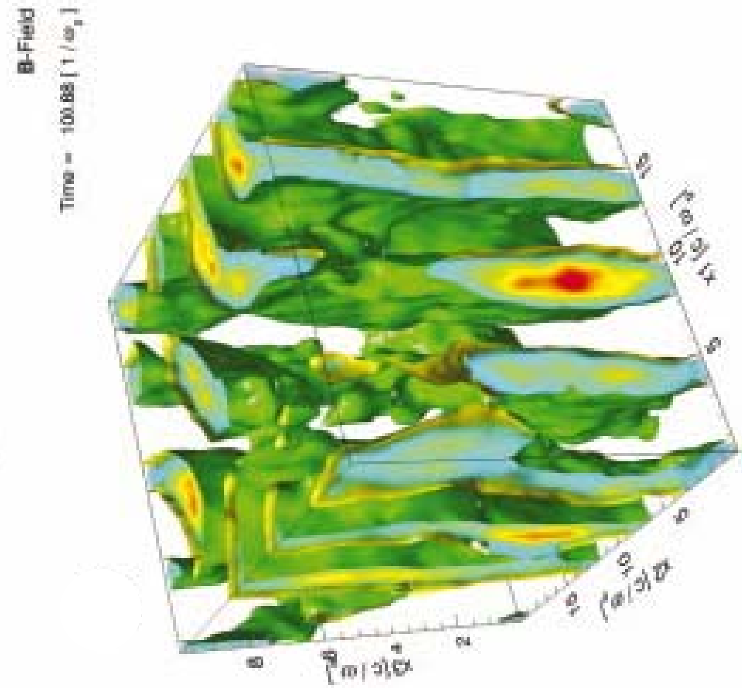
Collisionless shock wave in e--e+ plasma



Keshet, Katz, Spitkovsky, Waxman (2008)

3D Weibel instability in e$^-$ -e+ plasma





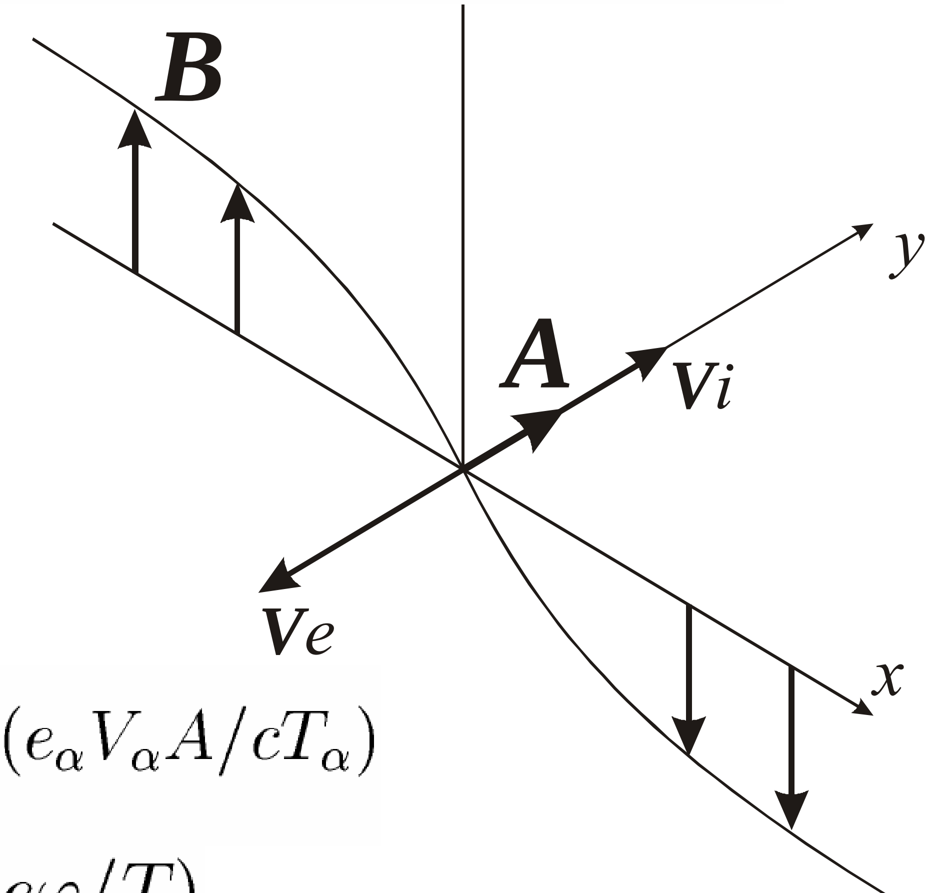
Magnetic field energy density for values of 15% of the maximum energy density. Results are shown slightly before saturation and in the quasi-static stage (ε_B~ 1%).

Fonseca, Silva et al (2003).

The Harris current sheet solution

$$f_{e,i} = \frac{N(x,z)}{(2\pi m_{e,i}T_{e,i})^{3/2}} \exp\left(\frac{-p_x^2 - (p_y - m_{e,i}V_{e,i})^2 - p_z^2}{2m_{e,i}T_{e,i}}\right)$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$



$$V_e, V_i = \mathrm{const}$$

$$V_e/T_e = V_i/T_i$$

$$\Delta A = -\frac{4\pi}{c}\sum_\alpha e_\alpha N_{0\alpha} V_\alpha \exp\left(e_\alpha V_\alpha A / cT_\alpha\right)$$

$$\Delta\varphi = 4\pi\rho, \qquad \rho \sim \exp(-e\varphi/T)$$

Balance of forces: $e_\alpha E_\alpha = T_\alpha \nabla N_\alpha$

Generalizations of Harris' solution

- Harris, 1962
- Fadeev et al., 1965
- Kan, 1973
- Channel, 1976
- Attico and Pegoraro, 1999
- Manakova et al., 2000
- Brittnacher and Whipple, 2002
- Schindler and Birn, 2002
- Mottez, 2003
- Yoon and Lui, 2005
- Zelenyi et al., 2006
- Suzuki and Shigeyama, 2008

Basic nonlinear equations describing stationary self-consistent current configurations in collisionless multicomponent plasma

$$\mathbf{p}\frac{\partial f_\alpha}{\partial \mathbf{r}} + \frac{e_\alpha}{c}[\mathbf{p} \times [\nabla \times \mathbf{A}]]\frac{\partial f_\alpha}{\partial \mathbf{p}} = 0$$

$$[\nabla \times [\nabla \times \mathbf{A}]] = \frac{4\pi}{c}\sum_\alpha e_\alpha \int f_\alpha \frac{\mathbf{p}}{m_\alpha \gamma_a} d^3\mathbf{p}$$

V.Ju. Martyanov, Vl.V. Kocharovsky, V.V. Kocharovsky et al., Proceedings of International Conference ``Frontiers of Nonlinear Physics", p. 647 (2005).
V.Ju. Martyanov, E.V. Derishev, V.V. Kocharovsky et al., AIP Conference proceedings **801**, 357 (2005).
V.Ju. Martyanov, Vl.V. Kocharovsky, V.V. Kocharovsky, JETP **107**, 1049 (2008)
Vl.V.Kocharovsky, V.V.Kocharovsky, V.Martyanov, Radiophys.Quant.Electr.**52**, n.2(2009)

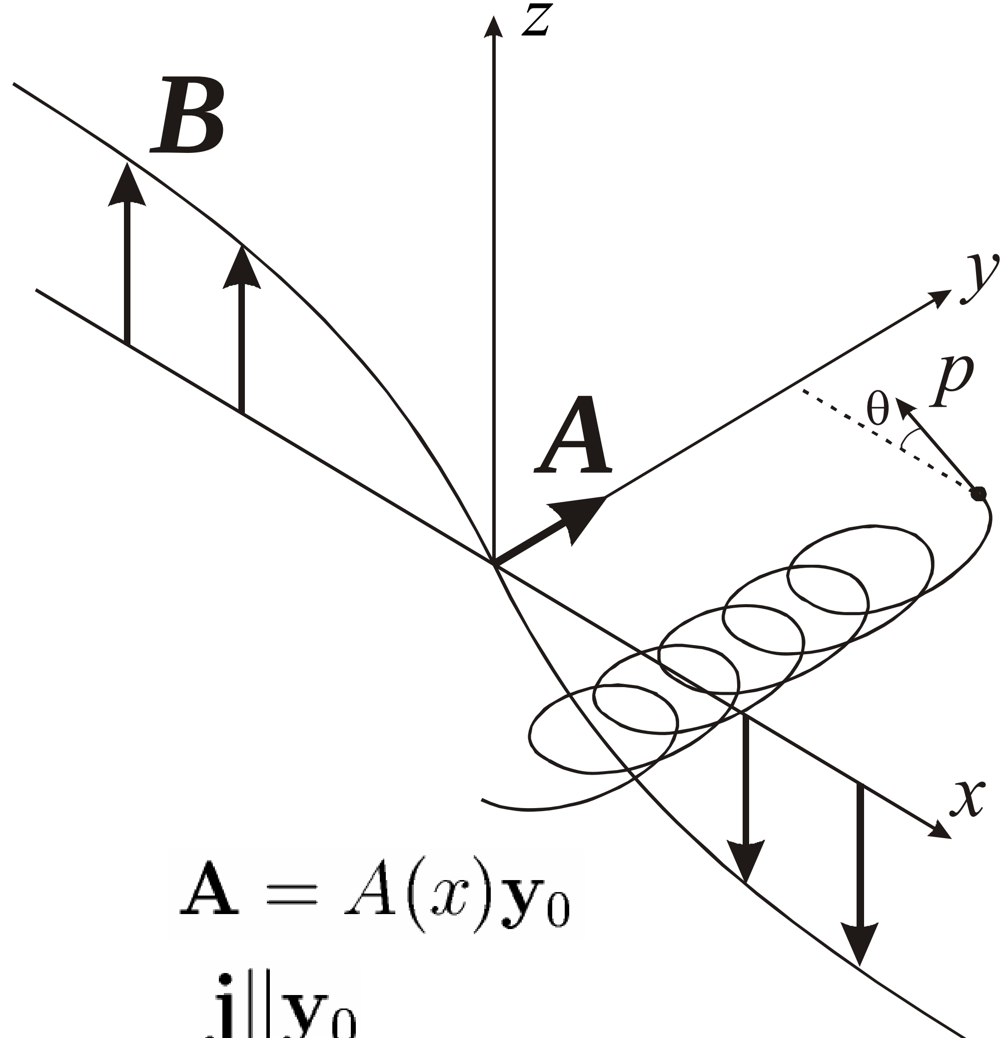
Integrals of particle motion

$$\mathcal{E} = c\sqrt{m^2c^2 + p^2}$$

$$P_y = p_y + \frac{e}{c}A_y$$

$$p_z$$

$$f = f(\mathcal{E}, P_y, p_z)$$



$$\mathbf{A} = A(x)\mathbf{y}_0$$

$$\mathbf{j} \| \mathbf{y}_0$$

Special case of PDF: cylindrical symmetry, $f = f(\mathcal{E}, P_y)$

Grad-Shafranov equation and PDF decomposition

$$f_\alpha\left(P_y,\mathcal{E}\right)=\sum_j f_{\alpha j}(\mathcal{E})\left(\frac{P_y}{m_\alpha c}\right)^j$$

$$\frac{\partial^2 A}{\partial x^2}+\frac{\partial^2 A}{\partial z^2}=-\frac{\partial U}{\partial A}$$

$$U=-8\pi^2 m_\alpha^2 c^3\sum_{j=0}^{\infty}\int f_{\alpha j}(\mathcal{E})Q_j\left(\frac{e_\alpha A}{m_\alpha c^2},\frac{p}{m_\alpha c}\right)\frac{p}{\gamma}dp$$

$$Q_j(A,p)=\frac{(A+p)^{j+2}\left[p(j+2)-A\right]+(A-p)^{j+2}\left[p(j+2)+A\right]}{(j+1)(j+2)(j+3)}$$

Harmonic solution of nonlinear problem (d=2)

$$\Delta_\perp A + k^2 A = 0$$

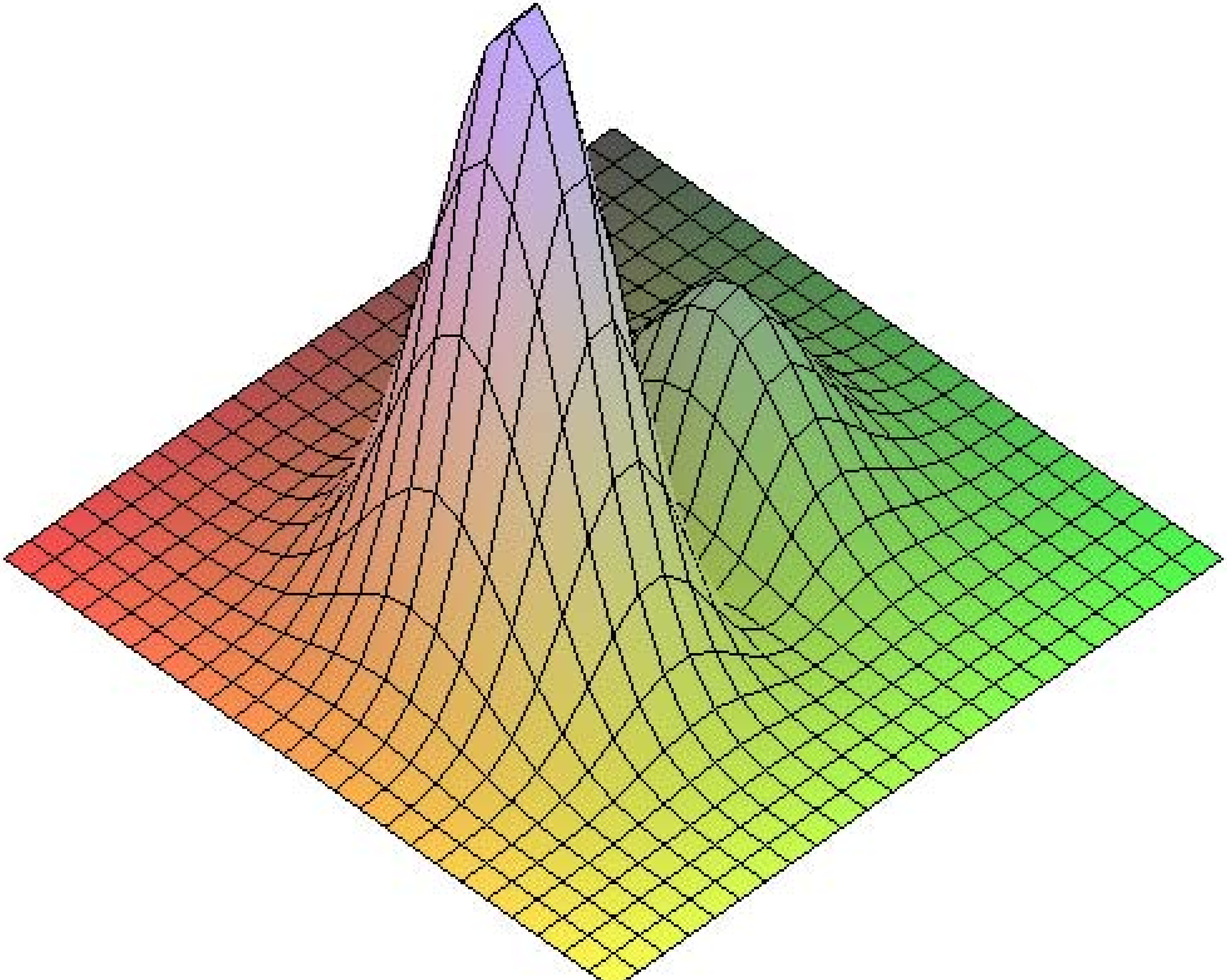
$$k^2 = \frac{32\pi^2}{3} \int f_2(\mathcal{E}) \frac{e^2 p^4}{m^3 c^4 \gamma} dp$$

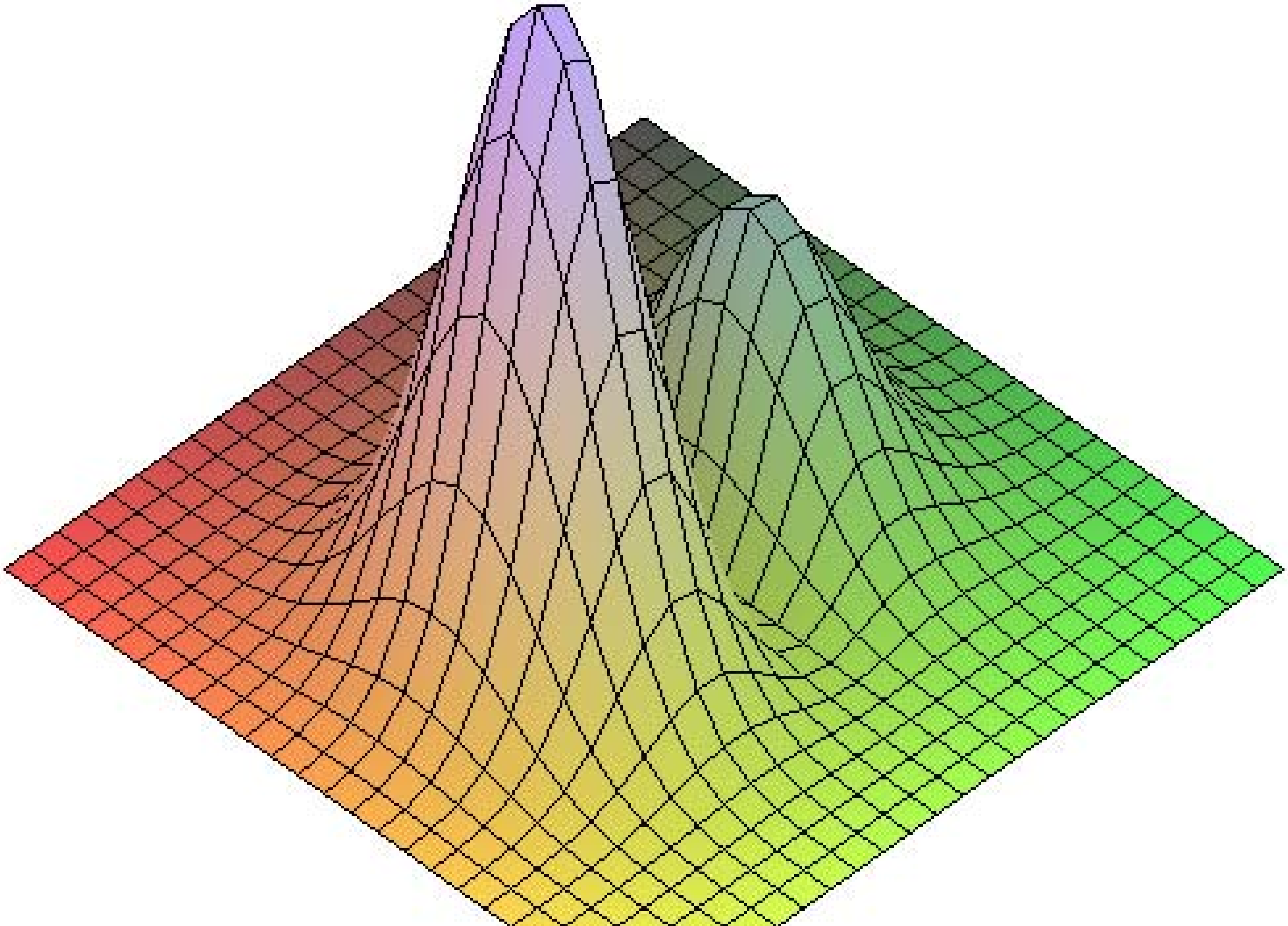
$$A = A_{\max} \cos(kx)$$

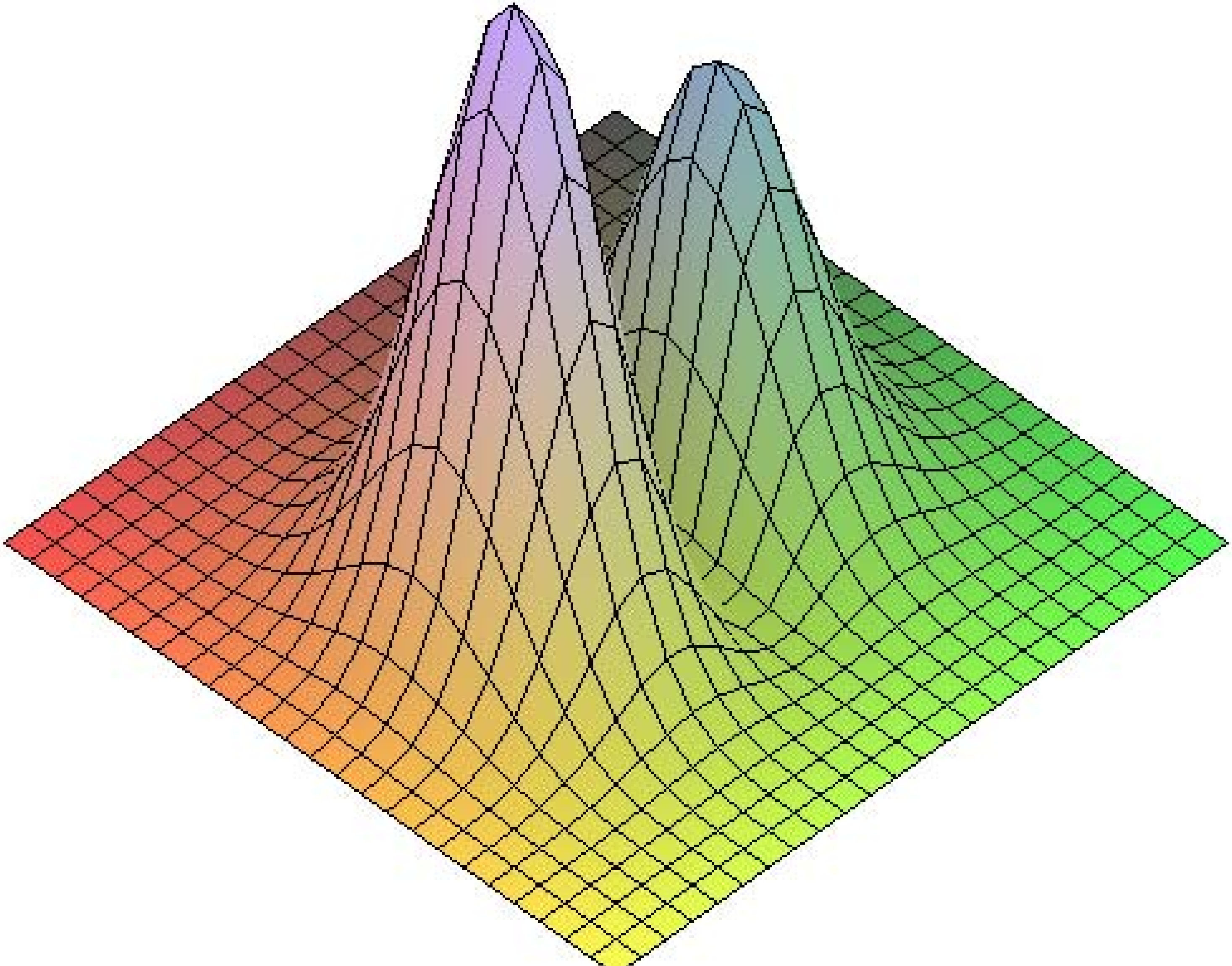
$$\frac{\langle W_B \rangle}{\langle W_e \rangle} = \frac{1}{3} \frac{\int f_2(\mathcal{E})(v^2/c^2)\gamma p^2 dp}{\int f_2(\mathcal{E})\gamma p^2 dp + \frac{2}{3}\int f_2(\mathcal{E}) p^2 c^2 \gamma p^2 \ dp / e^2 A_{\max}^2}$$

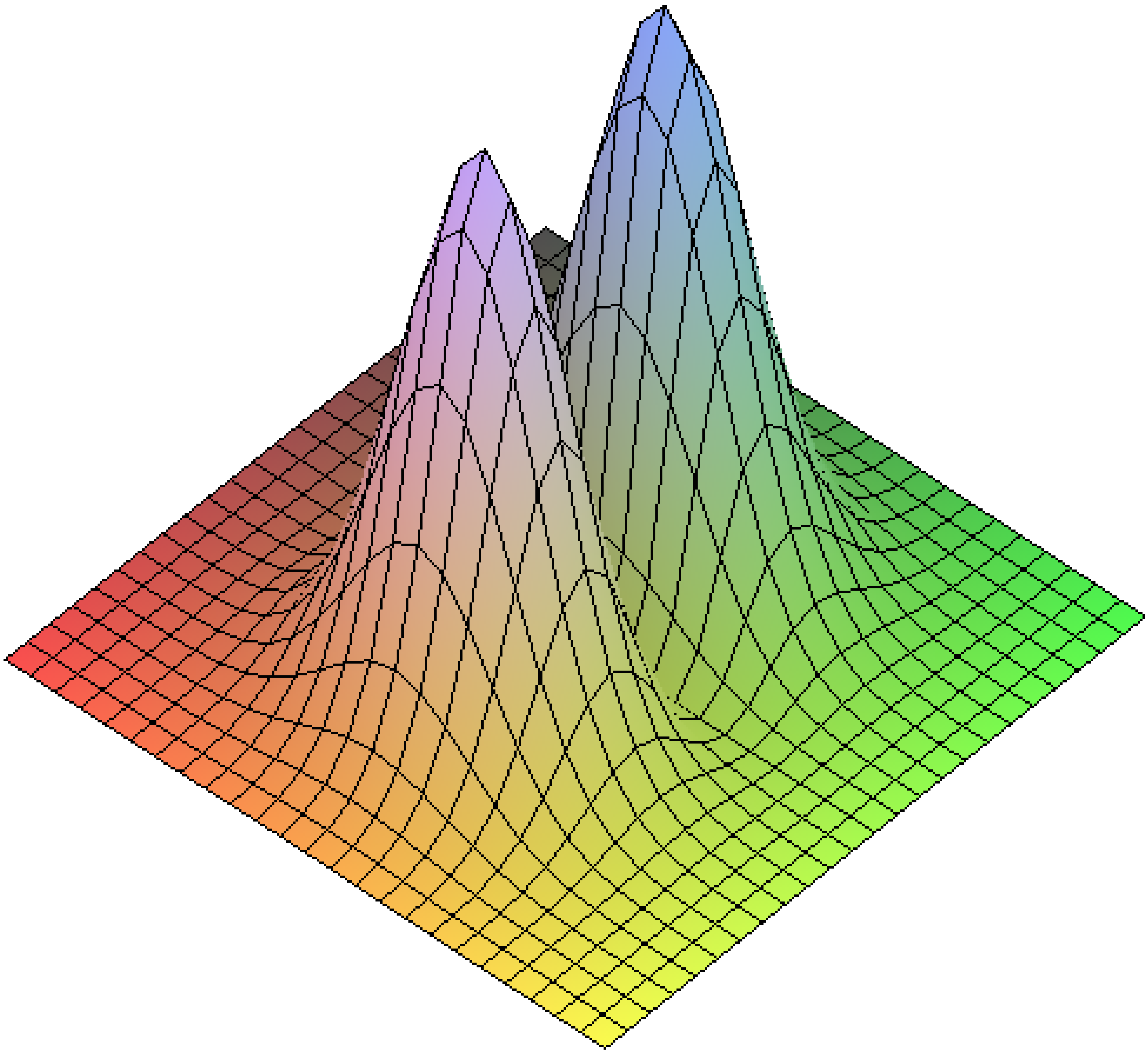
$$\frac{\langle W_B \rangle}{\langle W_e \rangle} = \frac{1}{3} \cdot \frac{1}{1 + \frac{2}{3}\frac{p^2 c^2}{e^2 A_{\max}^2}} \cdot \frac{v^2}{c^2} < \frac{1}{3}$$

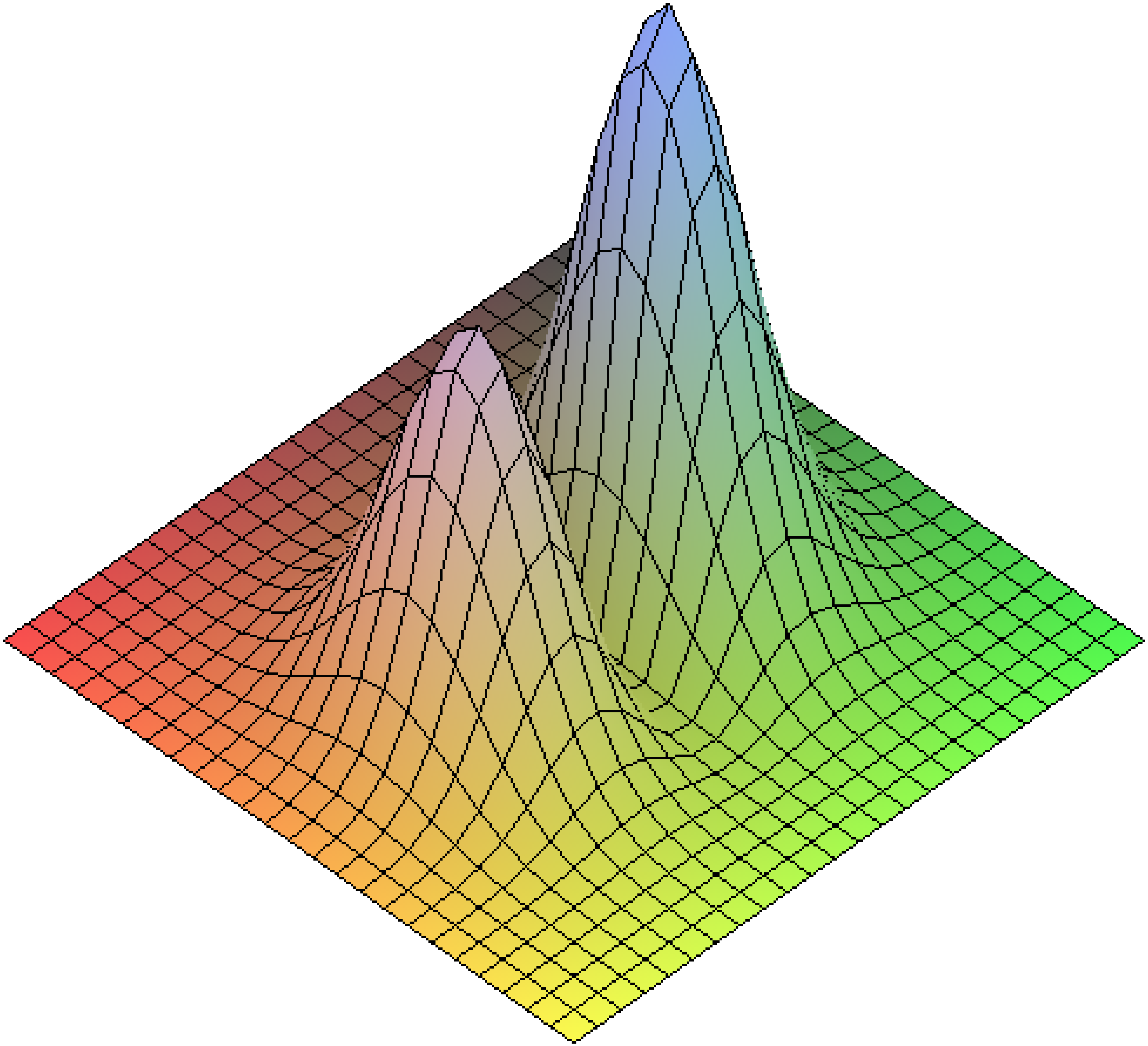
$$f(\mathcal{E}, P_y) \propto \exp(-\mathcal{E}/kT) P_y^2 \qquad A = A_{\max} \sin kx$$

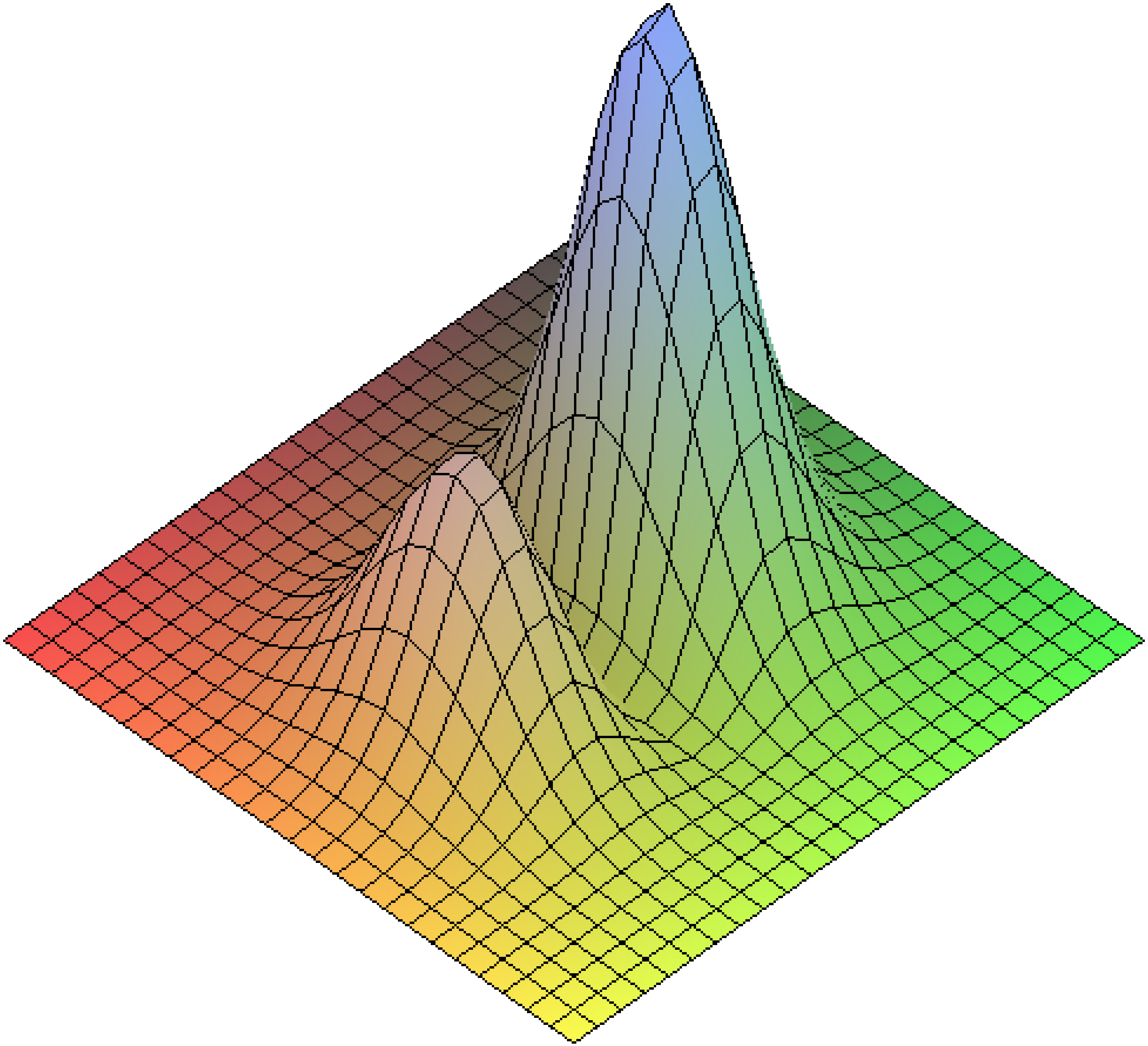


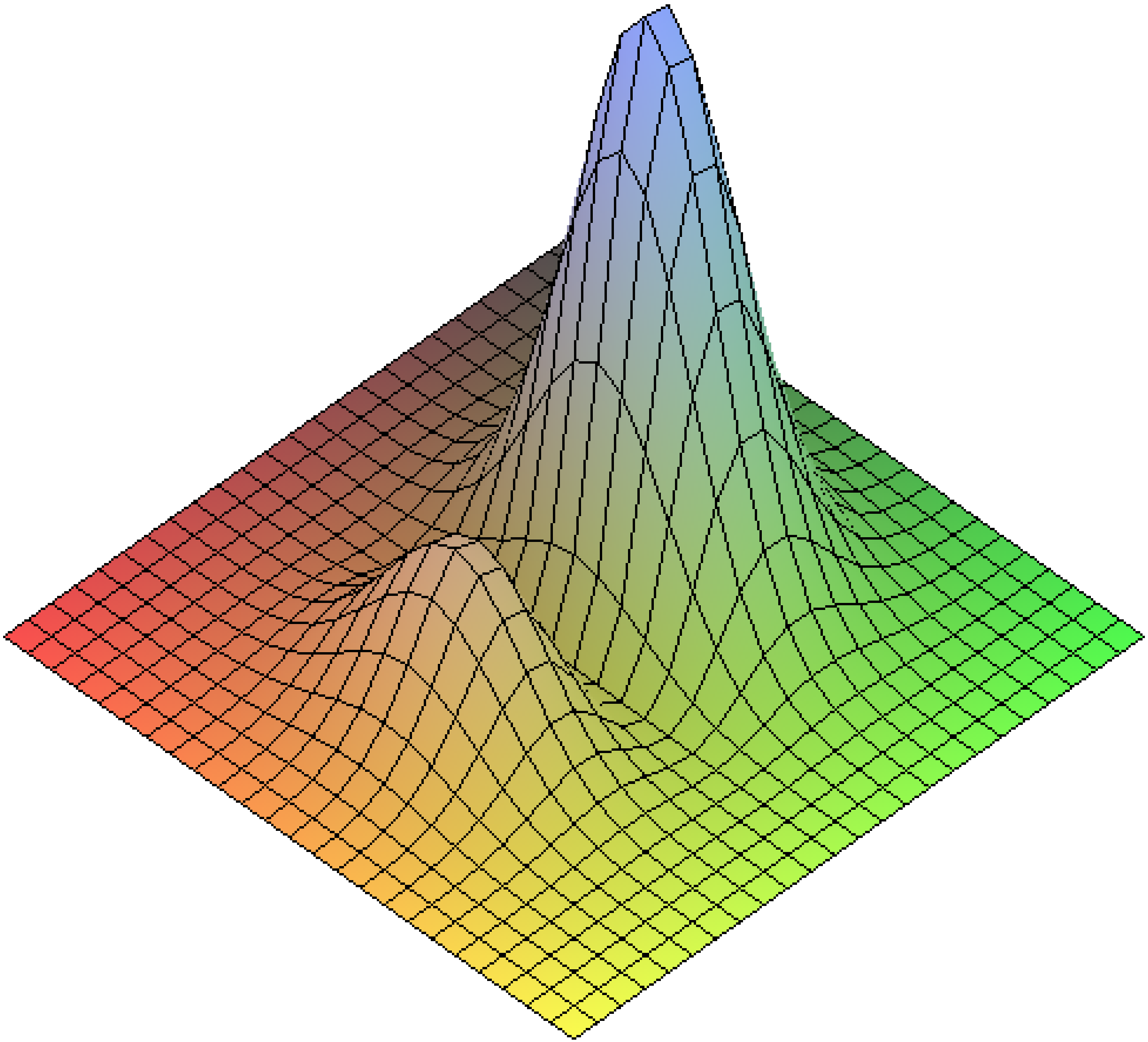






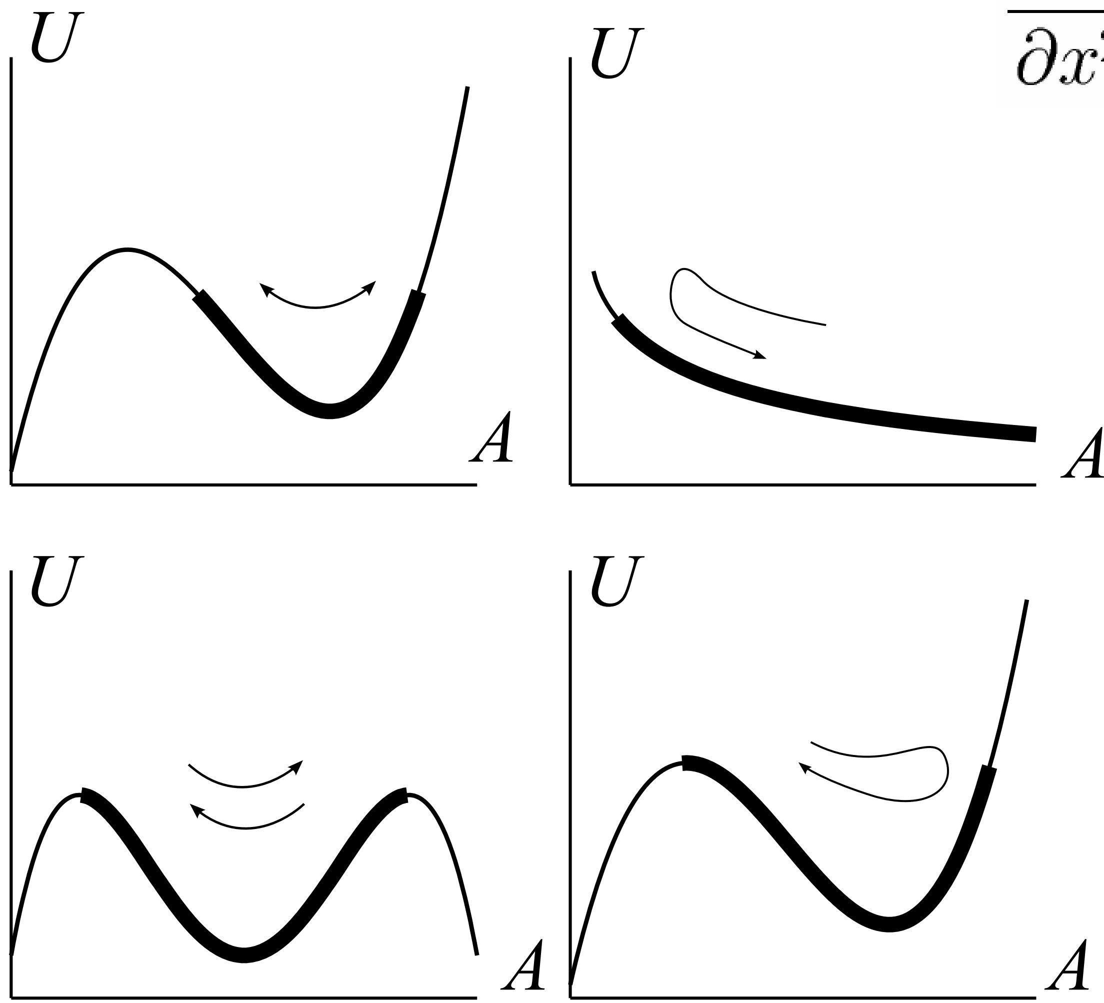






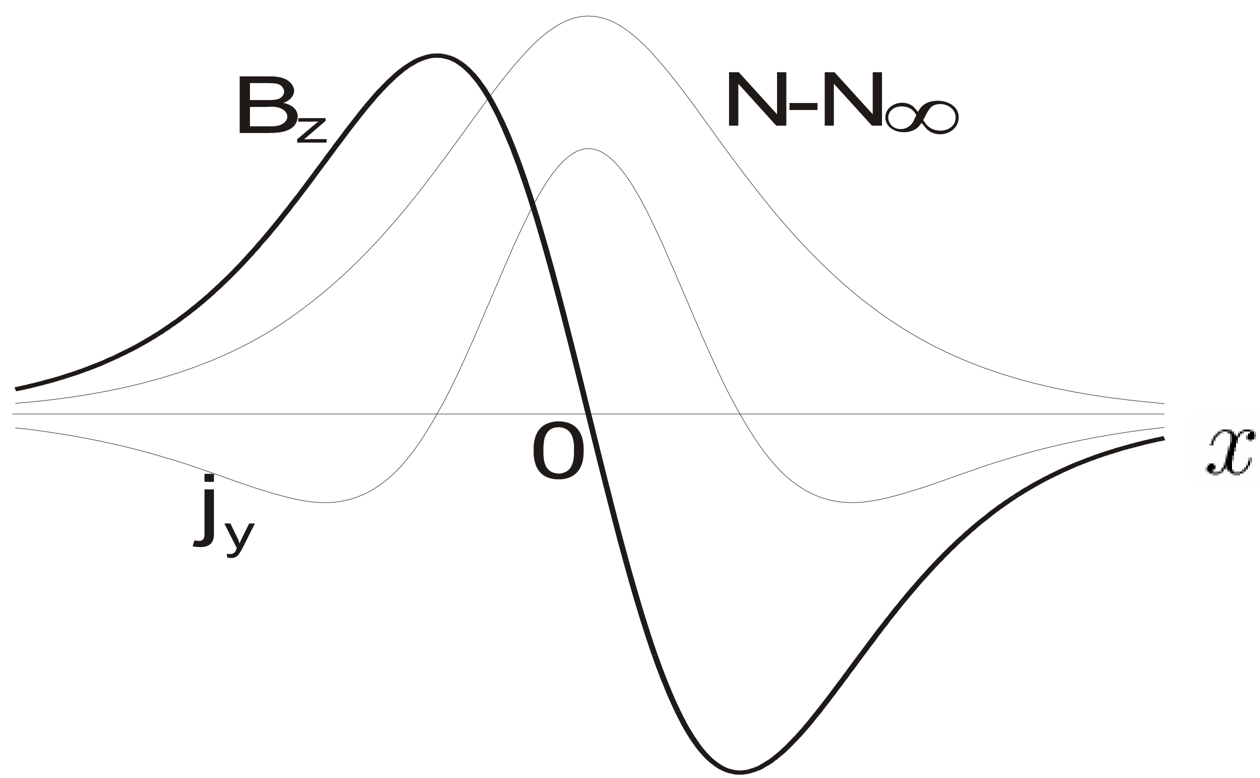
Grad-Shafranov potential and variety of solutions

$$\frac{\partial^2 A}{\partial x^2} = -\frac{\partial U}{\partial A}$$



Shielded current sheet (Taylor order d=3)

$$A = \frac{-mc^2 N_2}{eN_3}\,\mathrm{ch}^{-2}\left(\sqrt{-\frac{\pi e^2}{c\gamma_0}\frac{N_2 p_0^2}{(mc)^3}}\;x\right)$$



Shielded current sheet (Taylor order d=3)

$$A = \frac{-mc^2 N_2}{eN_3} \operatorname{ch}^{-2}\left(\sqrt{-\frac{\pi e^2}{c\gamma_0}\frac{N_2 p_0^2}{(mc)^3}}\ \ x\right)$$

$$B_z = \frac{2m_e c^2 N_2}{e_e N_3}\sqrt{-\frac{\pi e_e^2 N_2 p_0^2}{\gamma_0 m_e^3 c^4}} \operatorname{ch}^{-2}\left(\sqrt{-\frac{\pi e_e^2 N_2 p_0^2}{\gamma_0 m_e^3 c^4}}\ x\right) \tanh\left(\sqrt{-\frac{\pi e_e^2 N_2 p_0^2}{\gamma_0 m_e^3 c^4}}\ x\right),$$

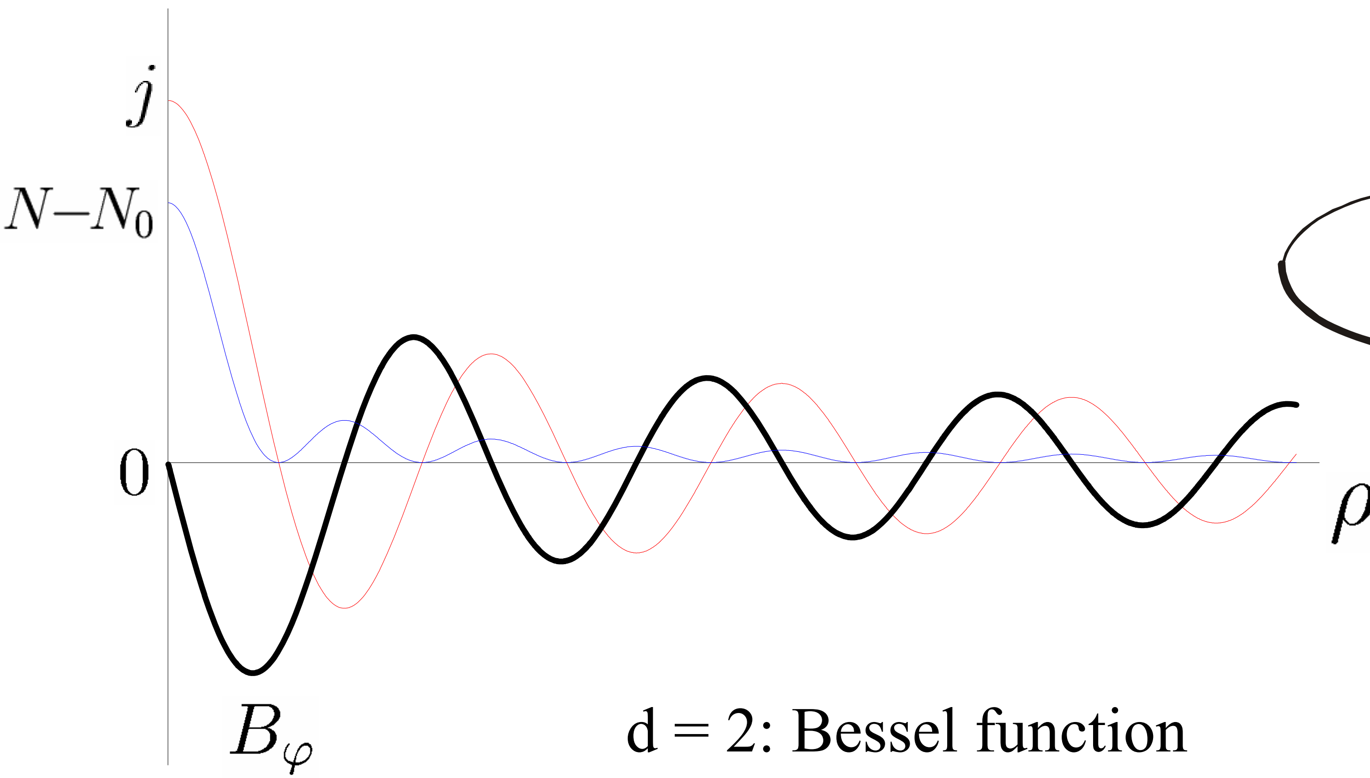
$$N_e = N_0 + \frac{N_2 p_0^2}{2m_e^2 c^2} + \frac{N_2 e_e^2 A_y^2}{m_e^2 c^4} + \frac{3N_3 p_0^2 e_e A_y}{4m_e^3 c^4} + \frac{N_3 e_e^3 A_y^3}{m_e^3 c^6},$$

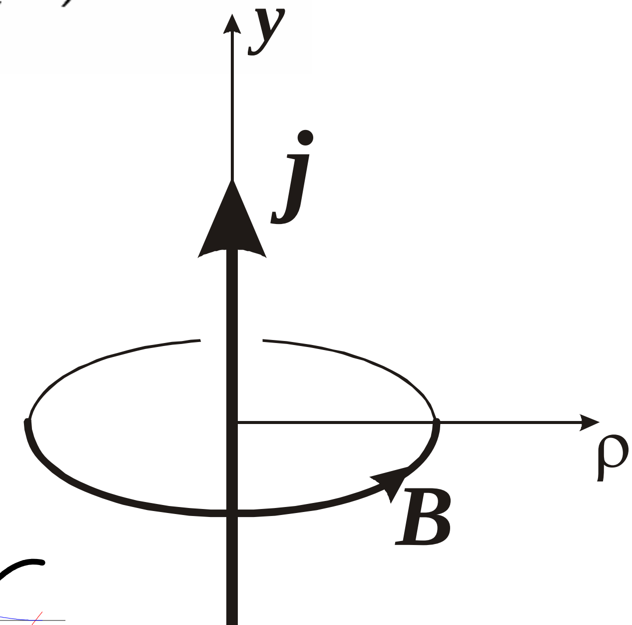
$$j_y = \frac{-N_2^2}{N_3}\frac{e_e c}{\gamma_0}\frac{p_0^2}{m_e^2 c^2} \operatorname{ch}^{-2}\left(\sqrt{-\frac{\pi e_e^2}{c\gamma_0}\frac{N_2 p_0^2}{(m_e c)^3}}\ \ x\right)\left[1 - \frac{3}{2}\operatorname{ch}^{-2}\left(\sqrt{-\frac{\pi e_e^2}{c\gamma_0}\frac{N_2 p_0^2}{(m_e c)^3}}\ \ x\right)\right],$$

$$B_{z\max} = -\frac{4\sqrt{3}}{9}\frac{m_e c^2 N_2}{e_e N_3}\sqrt{-\frac{\pi e_e^2}{c\gamma_0}\frac{N_2 p_0^2}{(m_e c)^3}}, \qquad j_{y\max} = \frac{N_2^2 e_e c p_0^2}{2N_3\gamma_0 (m_e c)^2}.$$

Cylindrical configurations (current filaments)

$$\frac{d^2A}{d\rho^2} + \frac{1}{\rho}\frac{dA}{d\rho} = -2\pi e \frac{p_0}{\gamma_0}\int_0^{2\pi} f_y\left(p\sin\theta + \frac{e}{c}A\right)\sin\theta d\theta$$

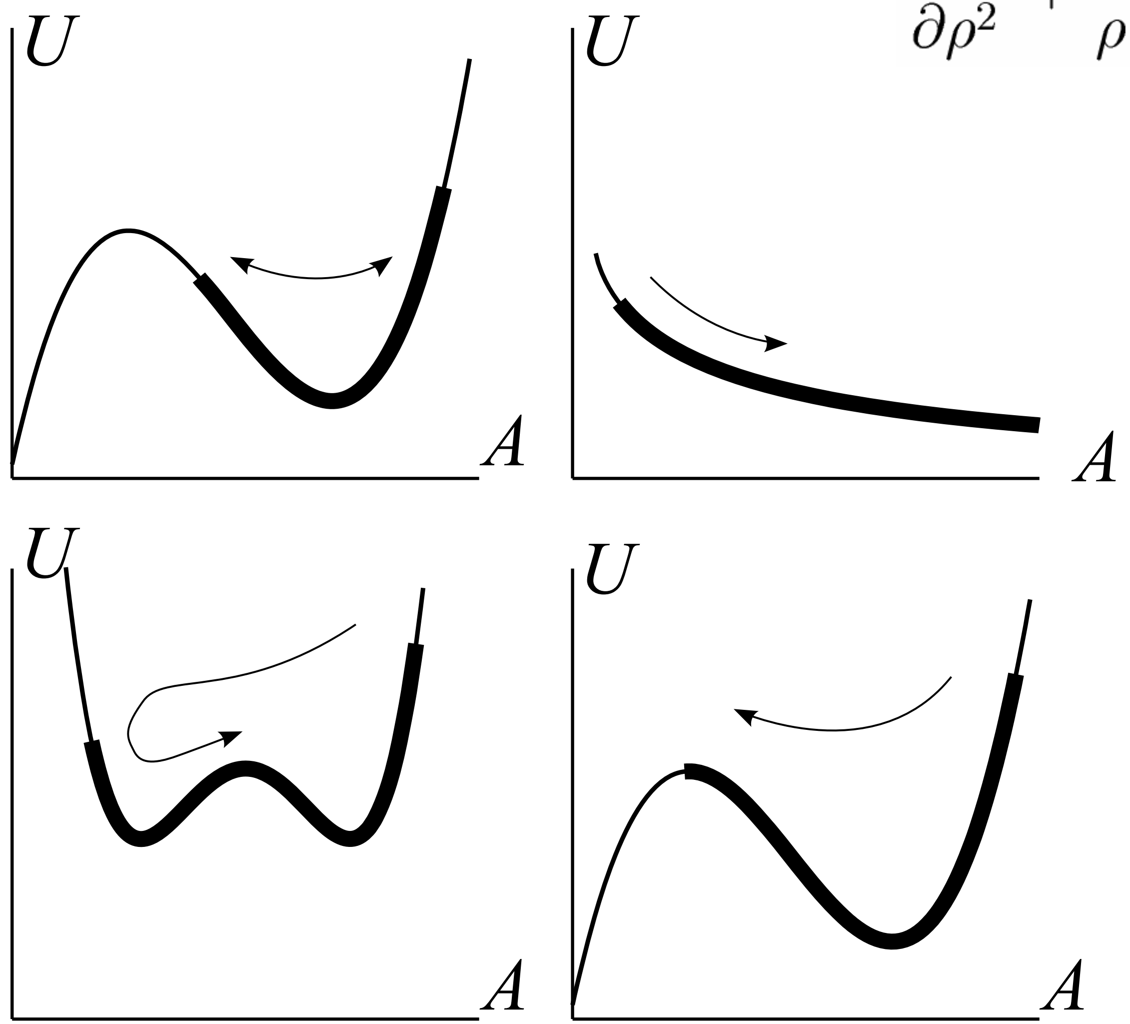




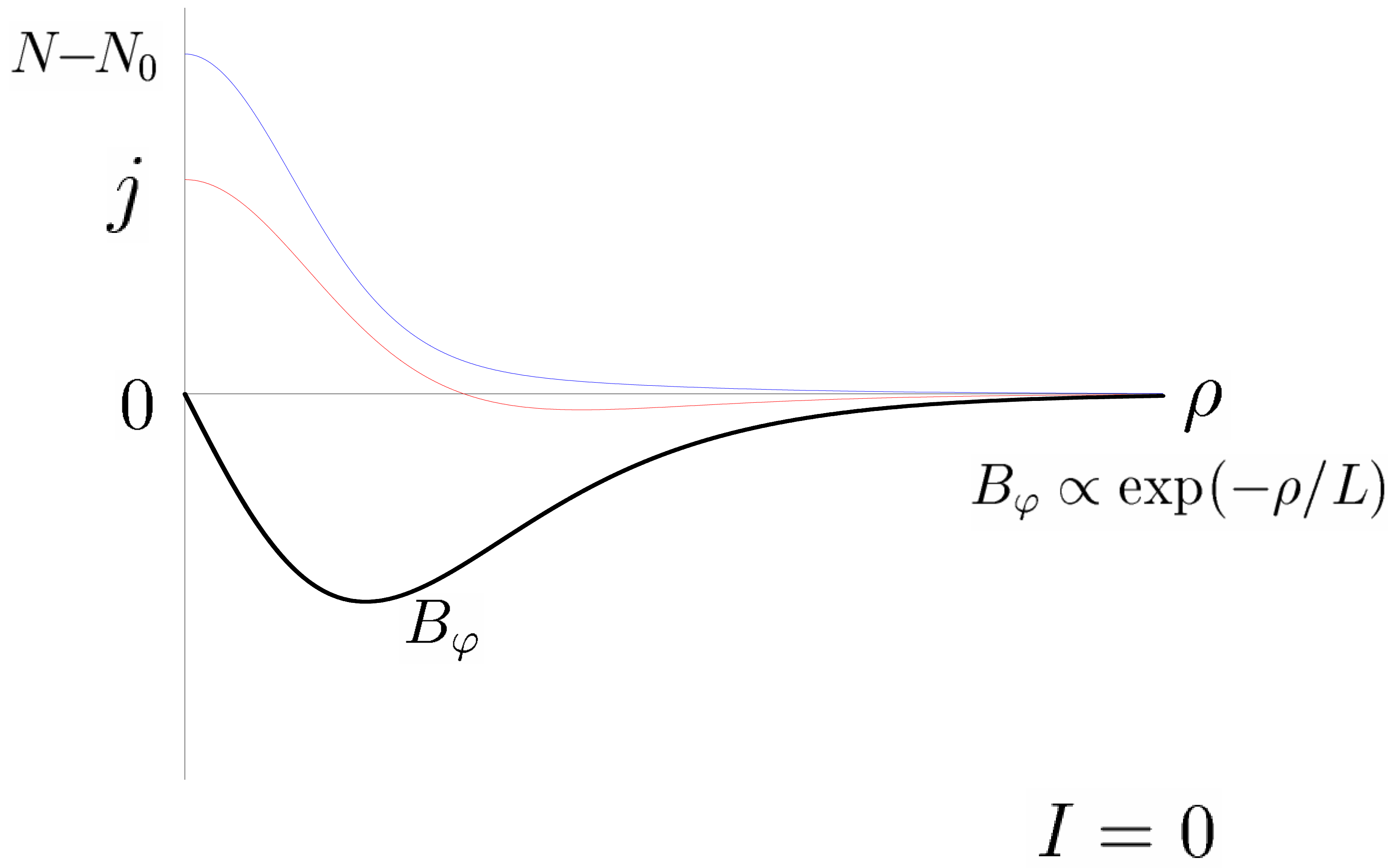
d = 2: Bessel function

Grad-Shafranov potential and effective viscous damping

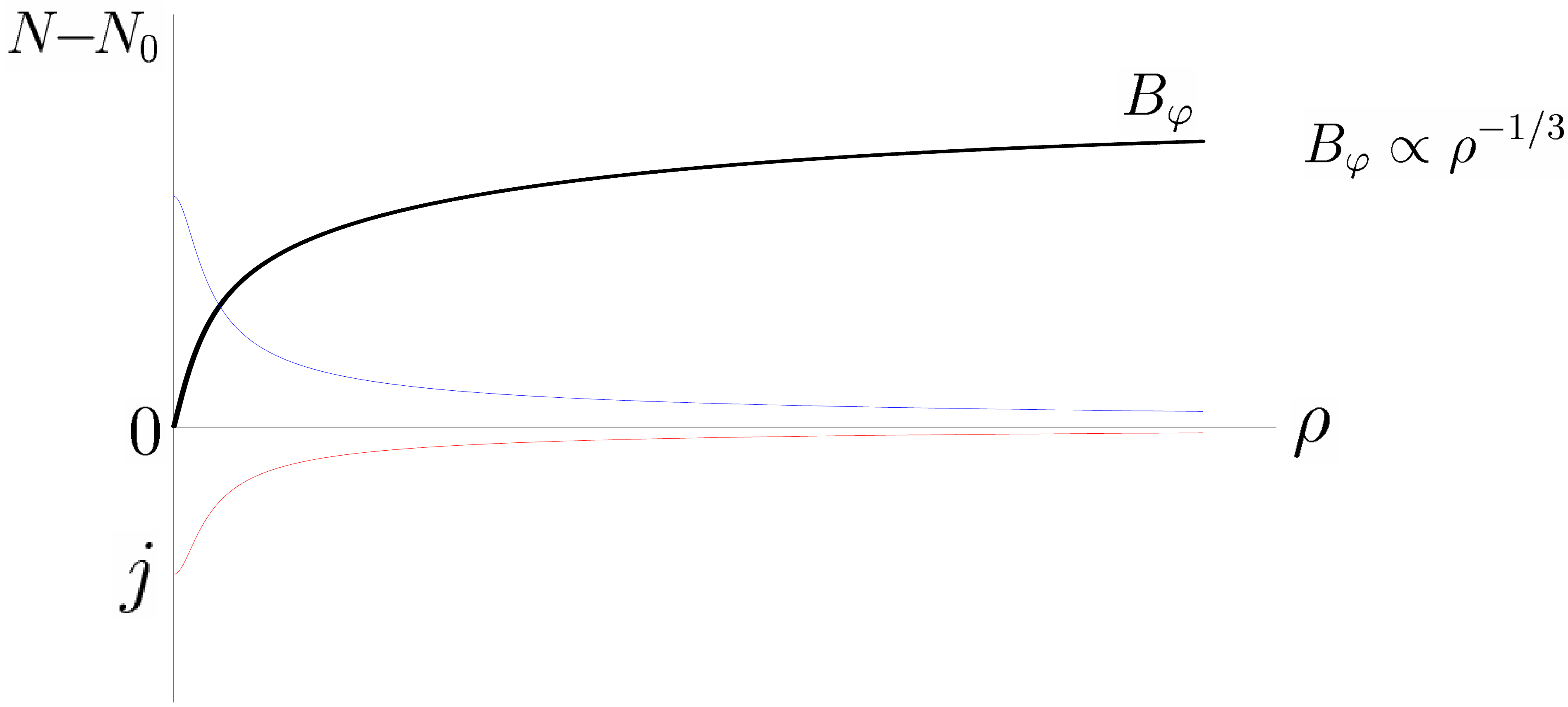
$$\frac{\partial^2 A}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial A}{\partial \rho} = -\frac{\partial U}{\partial A}$$



Shielded current filament (Taylor order d=3)

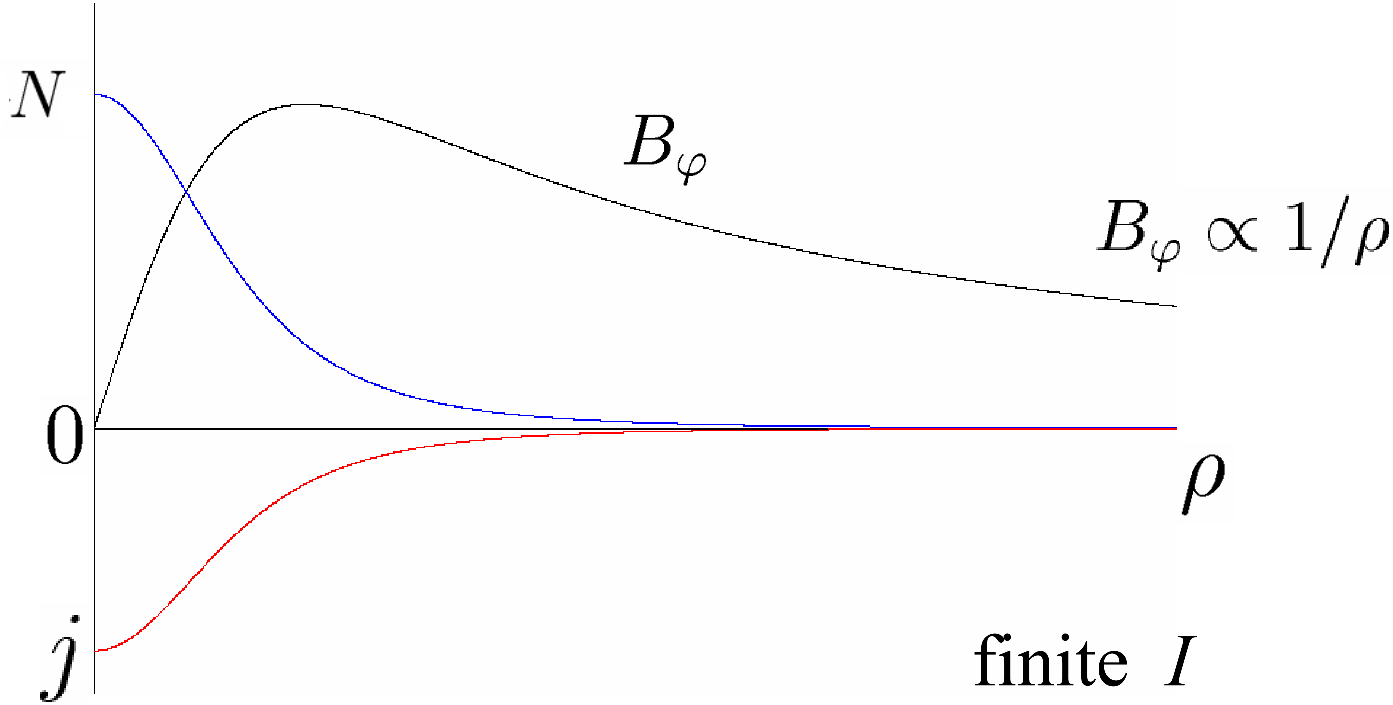


Single current filament (Taylor order d = -1)



$I = \infty$

Bennett pinch (PDF of exponential type)



$$A = -2A_0 \ln\left[1 + \frac{\alpha(x^2+y^2)}{8A_0}\right], \quad B = \frac{4A_0\alpha\rho}{8A_0 + \alpha\rho^2}, \quad N_j = N_j\max\left[1 + \frac{\alpha(x^2+y^2)}{8A_0}\right]^{-2}$$

Arbitrary PDF in generalization of Harris current sheet

$$\hat{F}_j(p, p_z + e_j A/c) = \hat{F}_j(p) \exp\left(\frac{cp_z/e_j + A}{A_0}\right) + \hat{f}_j(p)$$

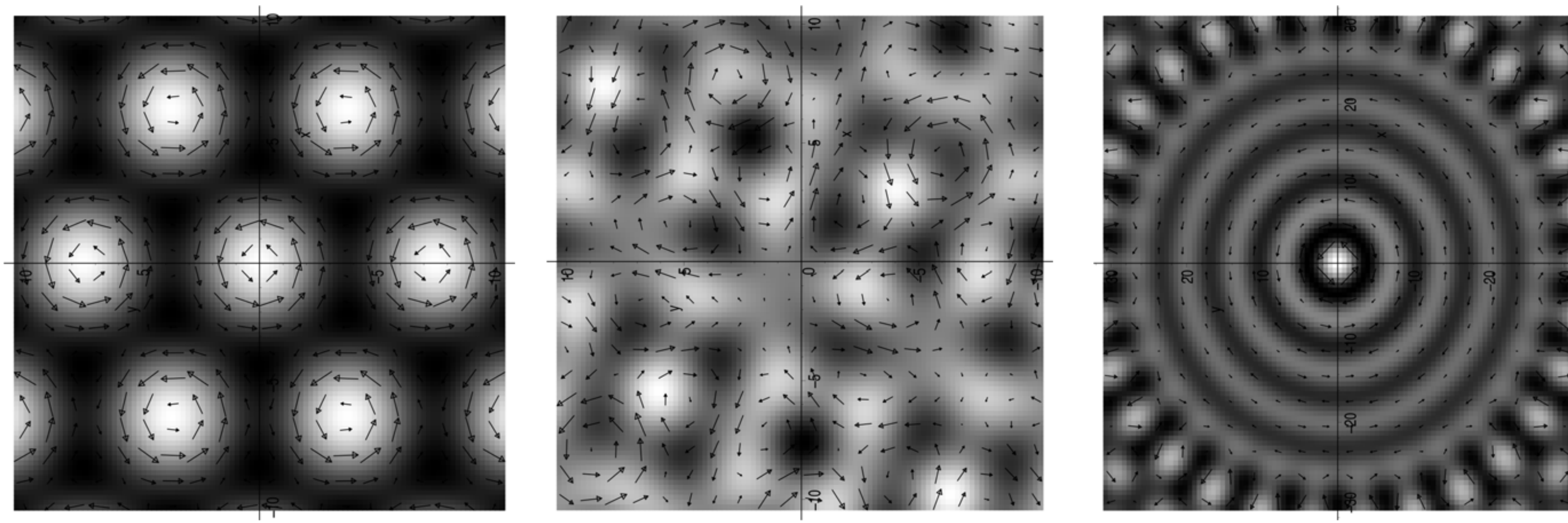
$$\Delta_{xy} A = -\alpha \exp\left(\frac{A}{A_0}\right)$$

$$\alpha = \sum_j \frac{8\pi^2 e_j^3 A_0^2}{m_j c^3} \int \left[\left(\frac{cp}{e_j A_0} - 1\right) \exp\left(\frac{cp}{e_j A_0}\right) + \left(\frac{cp}{e_j A_0} + 1\right) \exp\left(-\frac{cp}{e_j A_0}\right)\right] \frac{p}{\gamma_j} \hat{F}_j(p)\, dp$$

$$A = -2A_0 \ln\cosh\sqrt{\frac{\alpha}{2A_0}}\, x, \quad B_y = -\sqrt{2A_0\alpha}\tanh\sqrt{\frac{\alpha}{2A_0}}\, x$$

$$N_j = N_j \max\cosh^{-2}\sqrt{\frac{\alpha}{2A_0}} x$$

Two-dimensional current structures *j(x,z)* (*d*=2)



$$A = \sum_l A_l \cos(k\ x \cos\theta_l + k\ y \sin\theta_l + \varphi_l)$$

Scaling

$$\hat{p} = \alpha p, \hat{N} = \beta N, \hat{x} = x\sqrt{\hat{\gamma}_0/\gamma_0\beta}$$

$$\hat{B}_Z(x) = \alpha\sqrt{\frac{\beta\gamma_0}{\hat{\gamma}_0}}\; B_Z\left(x\sqrt{\frac{\beta\gamma_0}{\hat{\gamma}_0}}\right)$$

$$\frac{\langle W_B\rangle}{\langle W_e\rangle} = \frac{\langle B^2/8\pi\rangle}{\langle mc^2(\gamma-1)\rangle} \approx \text{const}$$

Kinetic features of self-consistent current structures

- L << r_H - most of the particles are not magnetically trapped ($I << I_A$)
- L >> r_H - the current is formed mainly by trapped particles ($I >> I_A$)

Degree of anisotropy is bounded by Taylor order *d*: $\frac{\langle p_y^2 \rangle}{\langle p_\perp^2 \rangle} < d$

Stability in the region where magnetic field vanishes:

Perturbations with ***E*** $\perp$ ***y*, *k*** || ***y*** can be unstable for high enough $\frac{\langle p_\perp^2 \rangle}{\langle p_y^2 \rangle}$

For d=4 perturbations with ***E*** $\perp$ ***y*, *k*** || ***y*** and with ***k*** $\perp$ ***y***, ***E*** || ***y*** are stable, if

$$\sum_\alpha \frac{e_\alpha^2}{m_\alpha}\left[5\int \frac{f_\alpha^2(\mathcal{E})}{\gamma_a}\left(\frac{p}{mc}\right)^2 p^2\,dp + 2\int \frac{f_\alpha^4(\mathcal{E})}{\gamma_a}\left(\frac{p}{mc}\right)^4 p^2\,dp\right] > 0$$

$$\sum_\alpha \frac{e_\alpha^2}{m_\alpha}\left[5\int \frac{f_\alpha^2(\mathcal{E})}{\gamma_a}\left(\frac{p}{mc}\right)^2 p^2\,dp + 6\int \frac{f_\alpha^4(\mathcal{E})}{\gamma_a}\left(\frac{p}{mc}\right)^4 p^2\,dp\right] < 0$$

Spectral features of synchrotron radiation

More complex spectra than single power-law (due to self-consistency of particle momentum distribution and magnetic field)

Simplest case of power-law particle energy distribution

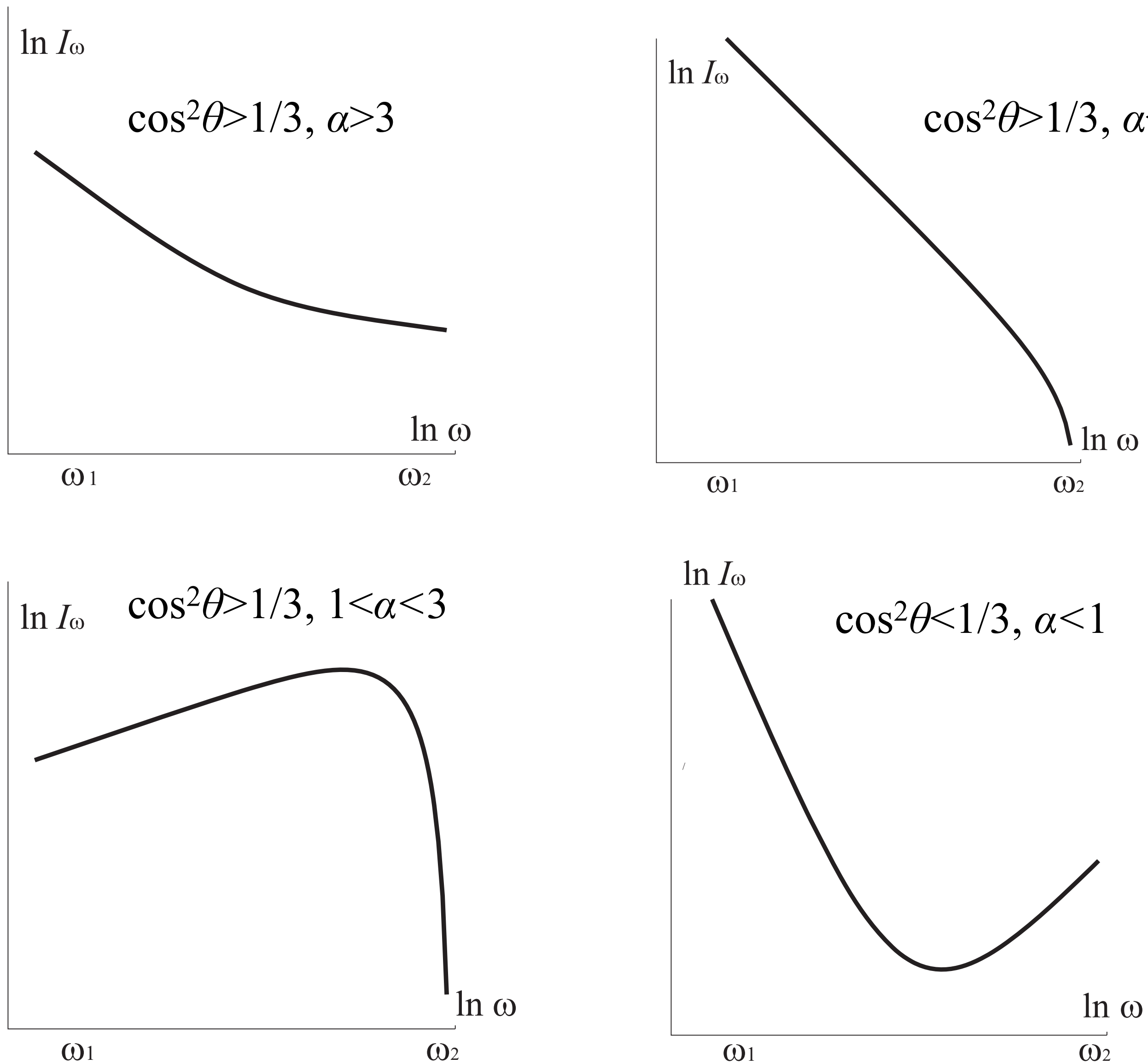
$$\int f_\alpha p^2 d\Omega \propto \mathcal{E}^{-\alpha} \quad (\mathcal{E}_1 < \mathcal{E} < \mathcal{E}_2)$$

d = 2:

$$f_\alpha \propto p^{-\alpha-2}\left[\left(\frac{p\cos\theta + e_\alpha A_y/c}{m_\alpha c}\right)^2 - \frac{p^2}{3m_\alpha^2 c^2} + \text{const}\right]$$

$$I_\omega = a_1\omega^{-(\alpha-3)/2}\left(\cos^2\theta - \frac{1}{3}\right) + a_2\omega^{-(\alpha-1)/2}$$

Double power-law, single- or double-humped spectra.
For chaotic ensemble of sheets or filaments PDF anisotropy vanishes, and synchrotron emission anisotropy remains


ln I_ω
$\cos^2\theta > 1/3$, $\alpha > 3$
ln ω
ω_1
ω_2
ln I_ω
$\cos^2\theta > 1/3$, $\alpha < 3$
ln ω
ω_1
ω_2
ln I_ω
$\cos^2\theta > 1/3$, $1 < \alpha < 3$
ln ω
ω_1
ω_2
ln I_ω
$\cos^2\theta < 1/3$, $\alpha < 1$
ln ω
ω_1
ω_2

Conclusions

- Closed analytical form of the nonlinear Grad-Shafranov equation is obtained on the basis of Taylor decomposition of current density in collisionless relativistic plasma

- As a result, exact solutions of Vlasov-Maxwell equations are found, describing a broad variety of self-consistent current filaments and sheets with arbitrary energy PDF

- Various properties of current filaments and sheets are investigated, including magnetic energy content, gyroradius to thickness ratio, PDF anisotropy, and synchrotron radiation

- The approach presented opens the possibility to build up an advanced theory of long-living filament-current turbulence in nonequilibrium anisotropic plasmas, silimar to Langmuir one