The Self-Interaction Problem in Classical Electrodynamics of Even-Dimensional Spacetimes

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Introduction

Rearrangement of the initial degrees of freedom appearing in the Lagrangian is a salient manifestation of self-interaction in field theory. The term 'rearrangement' was first introduced by Umezawa

H. Umezawa (1965) **Dynamical rearrangement of symmetries. The Nambu–Heisenberg model** *Nuovo Cimento* A **40** 450

who looked at *spontaneous symmetry breaking* for presentation of advantages of this concept. The mechanism for rearranging *classical gauge fields* was further studied in

B. P. Kosyakov (1992) **Radiation in electrodynamics and Yang–Mills theory** *Sov. Phys.–Uspekhi* **35** 135
B. P. Kosyakov (1998) **Exact solutions in the Yang–Mills–Wong theory** *Phys. Rev.* D **57** 5032
B. P. Kosyakov (1999) **Exact solutions of classical electrodynamics and Yang–Mills–Wong theory in even-dimensional spacetime** *Theor. Math. Phys.* **119** 493; hep-th/0207217
B. Kosyakov (2007) ***Introduction to the Classical Theory of Particles and Fields*** Heidelberg: Springer

Introduction

What is the essence of this mechanism? While having unlimited freedom in choosing dynamical variables for describing a given field system, preference is normally given to those variables which are best suited for implementing ***fundamental symmetries***. However, some degrees of freedom so introduced are ***dynamically unstable***. This gives rise to assembling the initial degrees of freedom into new, ***stable modes***.

Introduction

For example, the Lagrangian of quantum chromodynamics is expressed in terms of ***quarks*** and ***gluons***. If a system with these degrees of freedom would exhibit open color, there appears to be no reason for maintaining this system stable. Quarks and gluons combine in color-neutral clusters, ***hadrons*** and ***glueballs***, in the cold phase, or they form a lump of color-neutral ***quark-gluon plasma*** in the hot phase.

One further example is the Maxwell–Lorentz theory which is initially formulated in terms of mechanical variables $z_\mu(s)$ describing world lines of bare charged particles and the electromagnetic vector potential $A_\mu(x)$. The retarded interaction between these degrees of freedom makes them unstable, causing their rearranging into new dynamical entities: ***dressed particles*** and ***radiation*.**

The simplest rearrangement is implemented on the Lagrangian level. For example, taking $\phi = \mu/\lambda + \chi$ in

$$L = \frac{1}{2}\left(\partial\phi\right)^2 + \frac{\mu^2}{2}\phi^2 - \frac{\lambda^2}{2}\phi^4$$

results in converting the initial tachyon mode ϕ into the stable oscillatory mode χ .

Introduction

However, such is not the case in the Maxwell–Lorentz theory. The rearrangement of the initial degrees of freedom into dressed particles and radiation is impossible to achieve by a mere change of variables in the Lagrangian. We should employ the integration properties of the electromagnetic stress-energy tensor. If we adopt the retarded boundary condition, then the stress-energy tensor $\Theta^{\mu\nu}$ splits into two dynamically independent parts $\Theta^{\mu\nu} = \Theta_I^{\mu\nu} + \Theta_{II}^{\mu\nu}$. We define $R^\mu = x^\mu - z^\mu(s_{ret})$, the null vector drawn from the point on the world line where the signal was emitted $z^\mu(s_{ret})$ to the point x^μ where the signal was received. We then refer to the term $\Theta_{II}^{\mu\nu}$ as ***radiation*** if $\Theta_{II}^{\mu\nu}$ *propagates along the future light cone*, $R_\mu \Theta_{II}^{\mu\nu} = 0$, and varies as ρ^{-2} implying that *the same amount of energy-momentum flows through spheres of different radii*. In other words, $\Theta_{II}^{\mu\nu}$ is an ***integrable*** term whose ***integration over the future light cone is vanishing***. In contrast, $\Theta_I^{\mu\nu}$ is a ***nonintegrable*** term whose contribution to the ***integral over the future light cone is nonvanishing***, $R_\mu \Theta_I^{\mu\nu} \neq 0$.

In my talk, I will review the extension of this idea to flat even-dimensional spacetimes and dart a look at four-dimensional electrodynamics of massless charged particles.

Retarded field in even-dimensional world

Consider a single charged point particle moving along a timelike world line in flat spacetime of an arbitrary even dimension $d = 2n$, $n = 1,2, \ldots$ Our prime interest is with d in the range from $d = 2$ to $d = 10$. The world line is regarded as a smooth function of the proper time s. We ***suppose*** that ***the Maxwell-Lorentz electrodynamics is still valid***, that is, the field sector is given by

$$L = -\frac{1}{4\Omega_{d-2}} F_{\mu\nu} F^{\mu\nu} - A_\mu j^\mu,$$

$$j^\mu(x) = e\int_{-\infty}^{\infty} ds\; v^\mu(s)\; \delta^d[x - z(s)],$$

and the retarded boundary condition is imposed on the vector potential A^μ. Here, Ω_{d-2} is the area of the unit (d-2)-sphere, $v^\mu = \dot{z}^\mu = dz^\mu / ds$ is the d-velocity, and $\delta^d(R)$ is the d-dimensional Dirac delta-function. Close inspection of solutions to d-dimensional Maxwell's equations shows that the (*suitably normalized*) retarded field strength generated by a point charge living in a $2n$-dimensional world is expressed in terms of the retarded vector potentials due to this charge in $2m$-dimensional worlds nearby.

Those relationships read:

$$
\begin{aligned}
\mathcal{F}^{(2)} &= -\mathcal{A}^{(2)} \wedge \mathcal{A}^{(4)}, \\
\mathcal{F}^{(4)} &= -\mathcal{A}^{(2)} \wedge \mathcal{A}^{(6)}, \\
\mathcal{F}^{(6)} &= -\mathcal{A}^{(2)} \wedge \mathcal{A}^{(8)} - \mathcal{A}^{(4)} \wedge \mathcal{A}^{(6)}, \\
\mathcal{F}^{(8)} &= -\mathcal{A}^{(2)} \wedge \mathcal{A}^{(10)} - 2\mathcal{A}^{(4)} \wedge \mathcal{A}^{(8)}, \\
\mathcal{F}^{(10)} &= -\mathcal{A}^{(2)} \wedge \mathcal{A}^{(12)} - 3\mathcal{A}^{(4)} \wedge \mathcal{A}^{(10)} - 2\mathcal{A}^{(6)} \wedge \mathcal{A}^{(8)}.
\end{aligned}
$$

Retarded field in even-dimensional world

These relations between the retarded field strengths and vector potentials are found in:

B. Kosyakov (2008) **Electromagnetic radiation in even-dimensional spacetimes** *Int J Mod Phys*. **23** 4695

Recall, the *canonical representation* of a general 2-form $\omega^{(2n)}$ in spacetime of dimension $d = 2n$ is the sum of n exterior products of 1-forms:

$$\omega^{(2n)} = f_1 \wedge f_2 + \ldots + f_{2n-1} \wedge f_{2n}$$

In particular, $\omega^{(10)}$ is decomposed into the sum involving five terms. However, the retarded field strength $F^{(10)}$ contains only three exterior products, two less than the canonical representation.

Retarded field in even-dimensional world

A notable feature of those relationships is that the world line of the charge generating these field configurations is described by different numbers of the principal curvatures for different d. To be specific, the world line in $d = 2$ is a planar curve, specified solely by one parameter k, while that appearing in $d = 6$ is characterized by five essential parameters $\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5$. If we regard the world line in $d = 2n$ as the basic object, then both projections of this curve onto lower-dimensional spacetimes and its extensions to higher-dimensional spacetimes are rather arbitrary. However, this arbitrariness does not show itself in these relationships.

Retarded field in even-dimensional world

The ***advanced*** fields F_{adv} can be also represented as the sums of exterior products of 1-forms A_{adv}, whereas ***combinations*** $\alpha F_{ret} + \beta F_{adv}$, $\alpha\beta \neq 0$, ***are not***. Therefore, those relationships do not hold for field configurations satisfying the Stuckelberg–Feynman boundary condition. We thus see that the remarkably simple structures displayed in the relationships between the field strengths $F^{(2n)}$ and vector potentials $A^{(2m)}$ are ***inherently classical***.

Radiation

Apart from the overall numerical factor, the stress-energy tensor of the electromagnetic field takes the same form in any dimension,

$$\Theta_{\mu\nu} = \frac{1}{\Omega_{d-2}}\left(F_{\mu}^{\alpha}F_{\alpha\nu} + \frac{\eta_{\mu\nu}}{4}F^{\alpha\beta}F_{\alpha\beta}\right).$$

Since this tensor is to be integrated over $(d-1)$-dimensional spacelike surfaces, it is conveniently split into two parts, ***nonintegrable*** and ***integrable***, $\Theta^{\mu\nu} = \Theta_{I}^{\mu\nu} + \Theta_{II}^{\mu\nu}$
To identify the integrable part of the stress-energy tensor as the ***radiation***, we check the fulfilment of the following conditions:

(i) $\Theta_{I}^{\mu\nu}$ and $\Theta_{II}^{\mu\nu}$ are ***dynamically independent*** off the world line, that is,

$$\partial_{\mu}\Theta_{I}^{\mu\nu} = 0, \qquad \partial_{\mu}\Theta_{II}^{\mu\nu} = 0,$$

(ii) $\Theta_{II}^{\mu\nu}$ ***propagates along the future light cone*** drawn from the emission point,
(iii) the energy-momentum flux of $\Theta_{II}^{\mu\nu}$ ***goes as*** ρ^{2-d}

Radiation

One can show that the infrared behavior of $F^{(2n)}$ is controlled by $A^{(2)} \wedge A^{(2n+2)}$. More precisely, the leading long-distance term $A^{(2)} \wedge \overline{A}^{(2n+2)}$, where

$$\overline{A}_{\mu}^{(2n+2)} = \frac{1}{\rho^{n}} \lim_{\rho \to \infty} \rho^{n} A_{\mu}^{(2n+2)},$$

is responsible for the infrared properties of $F^{(2n)}$. It is then clear that $\Theta_{II}^{\mu\nu}$ is given by

$$\Theta_{II}^{\mu\nu} = -\frac{1}{N_{n}^{2}\Omega_{2n-2}} R^{\mu} R^{\nu} (\overline{A}^{(2n+2)})^{2}.$$

One can check that condition (i) holds for this construction. Fulfillment of conditions (ii) and (iii) is evident.
The ***radiation rate*** can be shown to become

$$\dot{P}_{\mu}^{(2n)} = -\frac{\rho^{2n-1}}{N_{n}^{2}\Omega_{2n-2}} \int d\Omega_{2n-2} R_{\mu} \left(\overline{A}^{(2n+2)}\right)^{2},$$

where $N_{n} = (2n-3)!!,\ n \geq 2.$

Radiation

These formulas make it clear that the radiation in 2n-dimensional spacetime is an infrared phenomenon stemming from the next even dimension $d = 2n + 2$. This conclusion is unlikely could be drawn from explicit expressions for the radiation rate in different dimensions $d = 2n$, such as

$$\dot{P}_\mu^{(4)} = -\frac{2}{3} a^2 v_\mu,$$

$$\dot{P}_\mu^{(6)} = \frac{1}{9}\frac{1}{5\cdot 7}\left\{4[16(a^2)^2 - 7\dot{a}^2]v_\mu - 3\cdot 5(a^2)^{\bullet} a_\mu + 6a^2(\dot{a}_\mu + a^2 v_\mu)\right\}.$$

Here, $a^\mu = \dot{v}^\mu = dv^\mu / ds$ is the d-acceleration, and the dot denotes differentiation with respect to the proper time s.

Equation of motion for a dressed particle

We begin with $d = 4$. The particle sector of the Maxwell-Lorentz theory is given by

$$-m_0 \int d\tau \, \sqrt{\dot{z} \cdot \dot{z}},$$

where m_0 stands for the mechanical mass of the bare particle.
Teitelboim showed that ***four-momentum balance on the world line*** takes the form

$$\dot{p}^\mu + \dot{P}^\mu = f^\mu.$$

C. Teitelboim (1970) **Splitting of Maxwell tensor: Radiation reaction without advanced fields**. *Phys. Rev.* D **1** 1572

Here, p^μ is the ***four-momentum attributed to the dressed particle***,

$$p^\mu = m v^\mu - \frac{2}{3} e^2 a^\mu,$$

with *m* being the renormalized mass,

$$m = \lim_{\varepsilon \to 0} [m_0(\varepsilon) + \frac{e^2}{2\varepsilon}].$$

The four-momentum balance equation tells us: the four-momentum $-f^\mu ds$, extracted from an external field during the period ds, is distributed between the four-momentum of the dressed particle dp^μ and the four-momentum carried away by radiation dP^μ.

Equation of motion for a dressed particle

The four-balance equation is identical to the ***Lorentz-Dirac equation***

$$ma^\mu - \frac{2}{3}e^2(\dot{a}^\mu + v^\mu a^2) = f^\mu .$$

In view of identities $v^2 = 1, \quad v \cdot a = 0, \quad v \cdot \dot{a} = -a^2$, the Lorentz-Dirac equation takes the form

$$\overset{v}{\perp}(\dot{p} - f) = 0,$$

where

$$\overset{v}{\perp}_{\mu\nu} = \eta_{\mu\nu} - \frac{\dot{z}_\mu \dot{z}_\nu}{\dot{z}^2}$$

is the projection operator on a hyperplane with normal $v^\mu = \dot{z}^\mu$.
This is just Newton's second law embedded in Minkowski space. We see that a dressed particle is an object with four-momentum $p^\mu = mv^\mu - (2/3)\, e^2 a^\mu$, whose behavior is governed by Newton's second law. The structure of this equation makes it clear that a dressed particle experiences only an external force f^μ . This equation contains no term through which the dressed particle interacts with itself.

In *d* = 6, the situation is ***essentially*** the same. However, ***technically***, there are several complications.

Equation of motion for a dressed particle

To kill all divergences, the particle sector must contain an ***additional term***,

$$-\int d\tau\,\gamma^{-1}\left\{m_0-\nu_0\left[\gamma\frac{d}{d\tau}\left(\gamma\frac{dz^\mu}{d\tau}\right)\right]^2\right\},\qquad\qquad \gamma^{-1}=\sqrt{\dot z\cdot\dot z}.$$

We then come to the six-momentum balance equation

$$\dot{\mathfrak{p}}^\mu+\dot P^\mu=f^\mu.$$

Here,

$$\mathfrak{p}^\mu=mv^\mu+\nu(2\dot a^\mu+3a^2v^\mu)+\frac{4}{45}e^2\left(\ddot a^\mu+\frac{16}{7}a^2a^\mu+2v^\mu\frac{da^2}{ds}\right)$$

is the ***six-momentum attributed to the dressed particle in the balance equation***, and *m* and ν are renormalized parameters involved in the action. This balance equation can be recast in the form

$$\overset{v}{\perp}(\dot p-f)=0.$$

However, the ***dressed particle's six-momentum in this equation*** p^μ ***is not identical*** to $\mathfrak{p}^\mu$,

Equation of motion for a dressed particle

$$p^{\mu} = mv^{\mu} + \nu(2\dot{a}^{\mu} + 3a^{2}v^{\mu}) + \frac{1}{9}e^{2}\left(\frac{4}{5}\ddot{a}^{\mu} + 2a^{2}a^{\mu} + v^{\mu}\,\frac{da^{2}}{ds}\right)$$

Massless charged particles

There are ***exceptional*** dynamical systems. Their initial degrees of freedom ***remain unchanged*** under switching-on the interaction. Examples are provided by classical electrodynamics of *massless* charged particles and the Yang–Mills–Wong theory of *massless* colored particles. These theories have one property in common, ***conformal invariance***. Owing to this symmetry, self-interaction does not create renormalization of mass.

Conventional wisdom says that an accelerated charge emits radiation. However, the net effect of radiation for a massless charged particle is compensated by an appropriate reparametrization of the world line. In other words, ***both radiation and dressing are absent*** from this theory.

Classical electrodynamics of massless charged particles do not experience rearranging. It is not a ***smooth limit*** of classical electrodynamics of massive charged particles. Conformal invariance has a dramatic effect on the picture as a whole: if this symmetry is broken, as in electrodynamics of massive charged particles, then self-interaction is different from that of conformally invariant systems.

Massless charged particles

Leptons of zero mass ***do not appear to exist***. Nevertheless, the interest in a point charge moving at the speed of light is sometimes expressed in the literature

W. Bonnor (1970) **Charge moving with the speed of light** *Nature* **225** 932
P. Dolan (1970) **Classical charged photon** *Nature* **227** 825

On the other hand, it is conceivable that ***quarks in quark–gluon plasma*** (QGP) reveal themselves as ***massless particles***. If a lump of QGP is formed in a collision of heavy ions, such as an Au + Au collision in the Relativistic Heavy Ion Collider (RHIC) at Brookhaven, then deconfinement triggers the chiral symmetry-restoring phase transition, whereby ***quarks become massless***. As the data from RHIC measurements suggest, the equation of state for QGP (pressure as a function of the energy density) above the transition temperature $T_c \sim 160$ MeV is approximately $p = (1\,/\,3)\varepsilon$, which is peculiar to a relativistic gas of massless particles.

Massless charged particles

It was demonstrated in

B. Kosyakov (2008) **Massless interacting particles** *J Phys A: Math Theor* **41** 465401

that integrating $\Theta^{\mu\nu}$ over the future light cone gives zero. This is the required result; otherwise we would invoke the renormalization of mass which is problematic in the theory free of dimensional parameters.

If we evaluate the 4-momentum associated with $\Theta^{\mu\nu}_{II}$, we then have

$$P^{\mu}_{II} = -\frac{2}{3}e^2\Lambda\int_{-\infty}^{\tau} d\tau\ \dot{z}^{\mu}\ddot{z}^2,$$

where Λ is a regularization parameter required from the regularization prescription to smear the so-called ray singularity.

Massless charged particles

A close inspection shows that the contribution of P^{μ}_{II} to the energy–momentum balance equation is absorbed by an appropriate reparametrization of the null curve. The net effect of P^{μ}_{II} is gauge removable,

$$\int_{\tau'}^{\tau''} d\tau \left(\dot{\eta}\dot{z}^{\mu} + \eta\ddot{z}^{\mu} - \frac{2}{3}e^{2}\Lambda\dddot{z}^{2}\dot{z}^{\mu} \right) = 0.$$

Indeed, the first and the last terms, with similar kinematical structures, cancel under a particular parametrization

$$d\tau = d\overline{\tau}\left[1 + \frac{1}{\overline{\eta}(\overline{\tau})}\frac{2}{3}e^{2}\Lambda\tau\int_{-\infty} d\sigma\ \dddot{z}^{2}(\sigma)\right],$$

with

$$\eta(\tau) = \overline{\eta}(\overline{\tau}) + \frac{2}{3}e^{2}\Lambda\int_{-\infty}^{\tau} d\sigma\ \dddot{z}^{2}(\sigma).$$

To summarize, the energy–momentum balance at a null world line amounts to the equation of motion for a bare particle. The initial degrees of freedom do not experience rearrangement, that is, dressed charged particles and radiation do not arise

Conclusion

The self-interaction treatment in the Maxwell-Lorentz electrodynamics relies heavily on three key notions: ***rearrangement*** of the initial degrees of freedom resulting in the occurrence of ***dressed*** particles and ***radiation***.

The ***retarded field strength*** $F_{\mu\nu}^{(2n)}$ due to a point charge in a $2n$-dimensional world can be algebraically expressed in terms of the ***retarded vector potentials*** $A_{\mu}^{(2m)}$ generated by this charge as if it were accommodated in $2m$-dimensional worlds nearby, $2 \le m \le n+1$ With this finding, the radiation part of the stress-energy tensor and the rate of radiated energy-momentum of the electromagnetic field takes a compact form.

We compared the properties of the ***equation of motion for a dressed particle*** in $d = 4$ and $d = 6$. This equation proves to be both energy-momentum balance on the world line and Newton's second law embedded in the $2n$-dimensional spacetime.

It was established that the $d = 4$ Maxwell-Lorentz theory of ***massless*** charged particles does not experience rearranging its initial degrees of freedom.
Massless charged particles ***do not radiate***.