Unfolded Description of AdS_4 Black Hole

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Plan

- Motivation. Unfolded formulation. Some important black hole properties

- Unfolded description of AdS_4 space-time

- Black hole as the deformation of AdS_4

- Integrating flow. Explicit coordinate-free BH metrics

- Conclusions

Motivation

Gravity s=2 $\Rightarrow$ Super-gravity $\Rightarrow$ Higher spin gauge theory (HS) $s = 0, 1/2, 1, \ldots, \infty$ **(Vasiliev)**

HS gauge theory:

Consistent theory of interacting massless fields $s = 0, 1/2, \ldots \infty$ **in** AdS **space-time. It is formulated at the level of equations of motion for all spins.**

String theory $\longrightarrow$ **unbroken phase** $\longrightarrow$ **HS theory ???**

GR black holes $\longrightarrow$ **SUGRA black holes** $\longrightarrow$ **HS black holes ???**

Unfolded formulation

- First order coordinate-free differential equations (differential forms formalism).

- Additional fields (infinitely many, in general) parameterize all on-shell derivatives of physical fields.

Example: free massless scalar in Minkowski space-time $\Box\phi(x) = 0$

Unfolding: $\phi, \quad \phi_\mu = \partial_\mu \phi, \quad \ldots, \quad \phi_{\mu_1 \ldots \mu_n} = \partial_{\mu_1} \phi_{\mu_2 \ldots \mu_n}$

Set of fields: $\phi, \quad \phi_\mu, \ldots \quad \phi_{\mu_1 \ldots \mu_n}$

Consistency condition: symmetric $\phi_{\mu_1 \ldots \mu_n}$

Equations of motion: ${\phi^\mu}_{\mu\mu_3 \ldots \mu_n} = 0$

Strategy

1. Write the unfolded equations for AdS_4 space-time and find objects relevant to black hole such as Kerr-Schild vectors

2. Find appropriate deformation of AdS_4 equations that leads to black hole

3. Find integrating flow with respect to deformation parameters mapping one system to another and try to integrate flow equations with AdS_4 initial data

Classical black hole properties

$d = 4$ Kerr solution in Minkowski space

1. $g_{\mu\nu} = \eta_{\mu\nu}(x) + M\varphi_{\mu\nu}(x)$ – no $O(M^2)$ terms $\Rightarrow$ Einstein equations reduce to **free** $s = 2$ Pauli-Fierz eqs.

$$\Box\varphi_{\mu\nu} - \partial_\mu\partial_\lambda\varphi^\lambda{}_\nu - \partial_\nu\partial_\lambda\varphi^\lambda{}_\mu = 0 \qquad (\varphi_\mu{}^\mu = 0)$$

2. Kerr-Schild form: $\varphi_{\mu\nu} = \frac{1}{U(x)}k_\mu(x)k_\nu(x)$, k^μ – Kerr-Schild vector

$$k_\mu k^\mu = 0\,, \qquad k^\mu D_\mu k_\nu = k^\mu \partial_\mu k_\nu = 0$$

3. BH provides Fronsdal fields $\phi_{\mu_1\ldots\mu_s} = \frac{M}{U}k_{\mu_1}\ldots k_{\mu_s}$

$\mathrm{s} = 0 \Rightarrow$	Klein-Gordon	$\Box\phi = 0$
$\mathrm{s} = 1 \Rightarrow$	Maxwell	$\Box\phi_\mu - \partial_\lambda\partial_\mu\phi^\lambda = 0$
$\mathrm{s} = 2 \Rightarrow$	Pauli-Fierz	$\Box\phi_{\mu\nu} - 2\partial_\lambda\partial_{(\mu}\phi_{\nu)}{}^\lambda = 0$
$\mathrm{s} = \mathrm{s} \Rightarrow$	Fronsdal	$\Box\phi_{\mu_1\ldots\phi_{\mu_s}} - s\partial_\lambda\partial_{(\mu_1}\phi^\lambda{}_{\mu_2\ldots\mu_s)} = 0$

4. Kerr-Schild presentation is also valid in AdS. Black hole massless fields $\phi_{\mu_1\ldots\mu_s}$ satisfies free massless spin-s equations in AdS_4.

Cartan formalism

Lorentz connection 1-form $\Omega_{[ab]} = \Omega_{[ab],\mu} dx^\mu$

Vierbein 1-form $h_a = h_{a,\mu} dx^\mu$, $a, b = 1, \ldots, 4$

$$R_{ab} = d\Omega_{ab} + \Omega_a{}^c \wedge \Omega_{cb} \qquad \text{Riemann 2-form}$$

$$R_a = dh_a + \Omega_a{}^b \wedge h_b = 0 \qquad \text{torsion 2-form}$$

$$g_{\mu\nu} = h_{a,\mu} h_{b,\nu} \eta^{ab}$$

Two-component spinor notation

$V^a \to V^{\alpha\dot{\alpha}}$, $\quad \alpha, \dot{\alpha} = 1, 2$, $\quad$ indices raised and lowered with $\quad \epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$

Maxwell $F_{ab} \to (F_{\alpha\alpha}, \bar{F}_{\dot{\alpha}\dot{\alpha}})$

Weyl $C_{abcd} \to (C_{\alpha(4)}, \bar{C}_{\dot{\alpha}(4)})$

Riemann $R_{ab} \to (R_{\alpha\alpha}, \bar{R}_{\dot{\alpha}\dot{\alpha}})$

AdS_4 space-time

Isometries: $o(3,2) \Rightarrow$ 10 Killing vectors.

Let V^a be an AdS_4 Killing vector:

$$D_a V_b + D_b V_a = 0\,, \qquad \kappa_{ab} = D_a V_b = -\kappa_{ba}$$

$$DV_a = \kappa_{ab} h^b\,, \quad D\kappa_{ab} = \lambda^2 (V_a h_b - V_b h_a) \longleftarrow \textbf{unfolded equations}$$

$$d\Omega_{ab} + \Omega_a{}^c \wedge \Omega_{cb} = \lambda^2 h_a \wedge h_b\,, \quad dh_a + \Omega_a{}^b \wedge h_b = 0 \longleftarrow \textbf{consistency}$$

Spinor form for AdS_4 equations:

$$DV_{\alpha\dot\alpha} = \frac{1}{2} h^\gamma{}_{\dot\alpha} \kappa_{\gamma\alpha} + \frac{1}{2} h_\alpha{}^{\dot\gamma} \bar\kappa_{\dot\alpha\dot\gamma}$$

$$D\kappa_{\alpha\alpha} = \lambda^2 h_\alpha{}^{\dot\gamma} V_{\alpha\dot\gamma}\,, \quad D\bar\kappa_{\dot\alpha\dot\alpha} = \lambda^2 h^\gamma{}_{\dot\alpha} V_{\gamma\dot\alpha}\,.$$

Properties of AdS system

1. AdS_4 covariant form

$$K_{AB} = K_{BA} = \begin{pmatrix} \lambda^{-1}\kappa_{\alpha\beta} & V_{\alpha\dot\beta} \\ V_{\beta\dot\alpha} & \lambda^{-1}\bar\kappa_{\dot\alpha\dot\beta} \end{pmatrix}, \qquad \Omega_{AB} = \Omega_{BA} = \begin{pmatrix} \Omega_{\alpha\beta} & -\lambda h_{\alpha\dot\beta} \\ -\lambda h_{\beta\dot\alpha} & \bar\Omega_{\dot\alpha\dot\beta} \end{pmatrix}$$

$$D_0 K_{AB} = 0\,, \qquad R_{0AB} = d\Omega_{AB} + \frac{1}{2}\Omega_A{}^C \wedge \Omega_{CB} = 0\,.$$

K_{AB} – AdS_4 **global symmetry parameter** $\rightarrow$ two Casimir invariants

$$C_2 = \frac{1}{4}K_{AB}K^{AB}\,, \qquad C_4 = \frac{1}{4}\mathrm{Tr}K^4.$$

2. The existence of source-free Maxwell tensor

$$F_{\alpha\alpha} = -\lambda^{-2}G^3\kappa_{\alpha\alpha},\,, \qquad G = \frac{\lambda^2}{\sqrt{-\kappa^2}} = (-F^2)^{1/4}$$

$F_{\alpha\alpha}$ and $\bar F_{\dot\alpha\dot\alpha}$ satisfy source free Maxwell equations and Bianchi identities

$$D_{\gamma\dot\alpha}F_\alpha{}^\gamma = 0\,, \qquad D_{\alpha\dot\gamma}\bar F_{\dot\alpha}{}^{\dot\gamma} = 0.$$

3. AdS_4 **Killing-Yano tensors**

$$Y_{\alpha\alpha} = iG^{-3}F_{\alpha\alpha}\,, \qquad *Y_{\alpha\alpha} = G^{-3}F_{\alpha\alpha}$$

in vector notation: $D_{(m}Y_{n)p} = 0, \quad D_{[m}*Y_{np]} = 0$

4. AdS_4 **unfolded equations in terms of Maxwell field**

$$DV_{\alpha\dot\alpha} = \frac{1}{2}\rho\, h^{\gamma}{}_{\dot\alpha}F_{\gamma\alpha} + \frac{1}{2}\bar\rho\, h_{\alpha}{}^{\dot\gamma}\bar F_{\dot\alpha\dot\gamma}\,, \qquad \rho = -\lambda^2 G^{-3},$$

$$DF_{\alpha\alpha} = -\frac{3}{2G}h^{\beta\dot\gamma}V^{\beta}{}_{\dot\gamma}F_{(\beta\beta}F_{\alpha\alpha)}.$$

5. Two first integrals

$$I_1 = V^2 - \frac{\lambda^2}{2}\left(\frac{1}{G^2} + \frac{1}{\bar G^2}\right)\,,$$

$$I_2 = \frac{1}{G^3\bar G^3}V^{\alpha\dot\alpha}V^{\alpha\dot\alpha}F_{\alpha\alpha}\bar F_{\dot\alpha\dot\alpha} - V^2\left(\frac{1}{G^2} + \frac{1}{\bar G^2}\right) + \frac{\lambda^2}{4}\left(\frac{1}{G^2} - \frac{1}{\bar G^2}\right)^2,$$

AdS_4 Casimir invariants: $\qquad C_2 = I_1\,, \qquad C_4 = I_1^2 + \lambda^2 I_2$

6. The existence of Kerr-Schild vectors

$$\Pi^{\pm}_{\alpha\beta} = \frac{1}{2}(\epsilon_{\alpha\beta} \pm \frac{1}{G^2} F_{\alpha\beta}), \qquad \bar{\Pi}^{\pm}_{\dot\alpha\dot\beta} = \frac{1}{2}(\epsilon_{\dot\alpha\dot\beta} \pm \frac{1}{\bar{G}^2} \bar{F}_{\dot\alpha\dot\beta}).$$

The projectors allow one to build null vectors for any given vector $V_{\alpha\dot\alpha}$

real: $$\xi^{+}_{\alpha\dot\alpha} = \Pi^{+\ \beta}_{\alpha} \bar{\Pi}^{+\ \dot\beta}_{\dot\alpha} V_{\beta\dot\beta}, \quad \xi^{-}_{\alpha\dot\alpha} = \Pi^{-\ \beta}_{\alpha} \bar{\Pi}^{-\ \dot\beta}_{\dot\alpha} V_{\beta\dot\beta}$$

complex: $$\xi^{+-}_{\alpha\dot\alpha} = \Pi^{+\ \beta}_{\alpha} \bar{\Pi}^{-\ \dot\beta}_{\dot\alpha} V_{\beta\dot\beta}, \quad \xi^{-+}_{\alpha\dot\alpha} = \Pi^{-\ \beta}_{\alpha} \bar{\Pi}^{+\ \dot\beta}_{\dot\alpha} V_{\beta\dot\beta}$$

Kerr-Schild vectors:

$$\boxed{\begin{array}{ll} k_{\alpha\dot\alpha} = \frac{2}{(V^+ V^-)} V^{-}_{\alpha\dot\alpha}, & n_{\alpha\dot\alpha} = \frac{2}{(V^+ V^-)} V^{+}_{\alpha\dot\alpha} \\ l^{-+}_{\alpha\dot\alpha} = \frac{2}{(V^{+-} V^{-+})} V^{-+}_{\alpha\dot\alpha}, & l^{+-}_{\alpha\dot\alpha} = \frac{2}{(V^{+-} V^{-+})} V^{+-}_{\alpha\dot\alpha} \end{array}}$$

$$e_{I,\alpha\dot\alpha} = (k_{\alpha\dot\alpha}, n_{\alpha\dot\alpha}, l^{+-}_{\alpha\dot\alpha}, l^{-+}_{\alpha\dot\alpha})$$

$$e_{I,\alpha\dot\alpha} e_{I,}{}^{\alpha\dot\alpha} = 0, \qquad e_{I,}{}^{\alpha\dot\alpha} D_{\alpha\dot\alpha} e_{I,\beta\dot\beta} = 0$$ **(no summation over I)**

Deformation of $AdS_4 \rightarrow$ BH unfolded system

Keep the same form of the unfolded equations

$$\mathcal{D}\mathcal{V}_{\alpha\dot\alpha} = \frac{1}{2}\rho\,\mathbf{h}^{\gamma}{}_{\dot\alpha}\mathcal{F}_{\gamma\alpha} + \frac{1}{2}\bar\rho\,\mathbf{h}_{\alpha}{}^{\dot\gamma}\bar{\mathcal{F}}_{\dot\alpha\dot\gamma}\,, \qquad \mathcal{D}\mathcal{F}_{\alpha\alpha} = -\frac{3}{2\mathcal{G}}\mathbf{h}^{\beta\dot\gamma}\mathcal{V}^{\beta}{}_{\dot\gamma}\mathcal{F}_{(\beta\beta}\mathcal{F}_{\alpha\alpha)}\,,$$

Unlike the AdS_4 case ρ is assumed to be arbitrary $\rho = \rho(\mathcal{G}, \bar{\mathcal{G}})$

Consistency: $\mathcal{D}^2 \sim \mathcal{R}\,, \mathcal{D}\mathcal{R} = 0$ fix $\rho(\mathcal{G}, \bar{\mathcal{G}})$ uniquely in the form

$$\rho(\mathcal{G}, \bar{\mathcal{G}}) = \mathcal{M} - \lambda^{\mathbf{2}}\mathcal{G}^{-\mathbf{3}} - \mathbf{q}\,\bar{\mathcal{G}}$$

Curvature 2-form is given by

$$\mathcal{R}_{\alpha\alpha} = \frac{\lambda^2}{2}\mathbf{H}_{\alpha\alpha} - \frac{3(\mathcal{M} - \mathbf{q}\,\bar{\mathcal{G}})}{4\mathcal{G}}\mathbf{H}^{\beta\beta}\mathcal{F}_{(\beta\beta}\mathcal{F}_{\alpha\alpha)} + \frac{\mathbf{q}}{4}\bar{\mathbf{H}}^{\dot\beta\dot\beta}\bar{\mathcal{F}}_{\dot\beta\dot\beta}\mathcal{F}_{\alpha\alpha}, \qquad \mathbf{H}^{\alpha\alpha} = h^{\alpha}{}_{\dot\alpha}\wedge h^{\alpha\dot\alpha}$$

Black hole Weyl tensor is of D-Petrov type i.e., $\mathbf{C}_{\alpha(4)} \sim \mathbf{F}_{\alpha\alpha}\mathbf{F}_{\alpha\alpha}$

Properties of the BH unfolded system

1. **AdS_4-Kerr-Newman-Taub-NUT black hole (rotated, EM and NUT-charged)**

   $\mathbf{M} = \mathbf{Re}\ \mathcal{M}$ – BH mass, $\mathbf{N} = \mathbf{Im}\ \mathcal{M}$ – NUT charge,

   $\mathbf{q} = e^2 + \mathrm{g}^2$ – sum of squared electric and magnetic charges

2. **Two integrals of motion**

$$\mathcal{I}_1 = \mathcal{V}^2 - \mathcal{M}\mathcal{G} - \bar{\mathcal{M}}\bar{\mathcal{G}} - \frac{\lambda^2}{2}\left(\frac{1}{\mathcal{G}^2} + \frac{1}{\bar{\mathcal{G}}^2}\right) + \mathbf{q}\,\mathcal{G}\bar{\mathcal{G}}\,,$$

$$\mathcal{I}_2 = \frac{1}{\mathcal{G}^3\bar{\mathcal{G}}^3}\mathcal{V}^{\alpha\dot{\alpha}}\mathcal{V}^{\alpha\dot{\alpha}}\mathcal{F}_{\alpha\alpha}\bar{\mathcal{F}}_{\dot{\alpha}\dot{\alpha}} - 2\left(\frac{\mathcal{M}}{\mathcal{G}} + \frac{\bar{\mathcal{M}}}{\bar{\mathcal{G}}}\right) - \mathcal{I}_1\left(\frac{1}{\mathcal{G}^2} + \frac{1}{\bar{\mathcal{G}}^2}\right) - \frac{\lambda^2}{4}\left(\frac{1}{\mathcal{G}^4} + \frac{1}{\bar{\mathcal{G}}^4}\right) - \frac{3\lambda^2}{2\mathcal{G}^2\bar{\mathcal{G}}^2}$$

3. **Inherits all AdS unfolded system properties**

   - **$\mathcal{F}_{\alpha\alpha}$ is a source-free Maxwell tensor**
   - **Killing projector construction allows to build two real and two complex Kerr-Schild vectors**

Integrating flow: $AdS_4 \Rightarrow$ BHUS

Let the deformation parameters $\psi = (\mathcal{M}, \bar{\mathcal{M}}, \mathbf{q})$ run $\Rightarrow$ one has corresponding flows $\frac{\partial}{\partial\psi}$.

Example: **due to completeness of $e_{I,\alpha\dot\alpha}$ basis and under gauge fixing**

$$\partial_{\mathcal{M}}\mathcal{V}_{\alpha\dot\alpha} = \sum_{I=1}^{4} \phi_I \hat{e}_{I,\alpha\dot\alpha}\,, \quad \partial_{\mathcal{M}}\mathbf{h}_{\alpha\dot\alpha} = \sum_{I=1}^{4} \phi_I \hat{e}_{I,\alpha\dot\alpha}\hat{e}_{I,\beta\dot\beta}\mathbf{h}^{\beta\dot\beta}.$$

Applying the integrability conditions to BHUS:

$$[d, \frac{\partial}{\partial\psi}] = [\frac{\partial}{\partial\psi}, \frac{\partial}{\partial\psi'}] = 0$$

we fix coefficients

$$\phi_1 = \frac{\mathcal{G}+\bar{\mathcal{G}}}{4}\alpha_1(r)\,, \qquad \phi_2 = \frac{\mathcal{G}+\bar{\mathcal{G}}}{4}\alpha_2(r)\,,$$

$$\phi_3 = \frac{\mathcal{G}-\bar{\mathcal{G}}}{4}\beta_1(y)\,, \qquad \phi_4 = \frac{\mathcal{G}-\bar{\mathcal{G}}}{4}\beta_2(y)$$

with “canonical” (**Frolov**) coordinates

$$r = \mathbf{Re}\frac{1}{\mathcal{G}}\,, \quad y = \mathbf{Im}\frac{1}{\mathcal{G}}\,, \quad \alpha_1 + \alpha_2 = \beta_1 + \beta_2 = 1.$$

Kerr-Schild vectors:

$$\hat{k}_{\alpha\dot{\alpha}} = k_{\alpha\dot{\alpha}} \left(\frac{\Delta_r}{\hat{\Delta}_r}\right)^{\alpha_2}, \quad \hat{n}_{\alpha\dot{\alpha}} = n_{\alpha\dot{\alpha}} \left(\frac{\Delta_r}{\hat{\Delta}_r}\right)^{\alpha_1},$$

$$\hat{l}^{+-}_{\alpha\dot{\alpha}} = l^{+-}_{\alpha\dot{\alpha}} \left(\frac{\Delta_y}{\hat{\Delta}_y}\right)^{\beta_2}, \quad \hat{l}^{-+}_{\alpha\dot{\alpha}} = l^{-+}_{\alpha\dot{\alpha}} \left(\frac{\Delta_y}{\hat{\Delta}_y}\right)^{\beta_1},$$

where

$$\hat{\Delta}_r = 2\mathrm{M}r + r^2(\lambda^2 r^2 + \mathcal{I}_1) + \frac{1}{2}(-\mathbf{q} + \frac{\mathcal{I}_2}{2})$$

$$\hat{\Delta}_y = 2\mathrm{N}y + y^2(\lambda^2 y^2 - \mathcal{I}_1) + \frac{1}{2}(\mathbf{q} + \frac{\mathcal{I}_2}{2}),$$

and

$$\Delta_{r,y} = \hat{\Delta}_{r,y}\Big|_{\mathcal{M},\overline{\mathcal{M}},\mathbf{q}=0}, \qquad e_{I,\alpha\dot{\alpha}} = \hat{e}_{I,\alpha\dot{\alpha}}\Big|_{\mathcal{M},\overline{\mathcal{M}},\mathbf{q}=0}.$$

Black hole metrics

- **General case** (Carter-Plebanski)

deformation parameters: $\mathbf{M}$ – black hole mass, $\mathbf{N}$ – NUT charge, $\mathbf{q} = e^2 + g^2$ – EM charges

$$ds^2 = ds_0^2 + \frac{2\mathbf{M}r - \mathbf{q}/2}{r^2 + y^2}(\alpha_1(r)K + \alpha_2(r)N)^2 - \frac{2\mathbf{N}y + \mathbf{q}/2}{r^2 + y^2}(\beta_1(y)L^{+-} + \beta_2(y)L^{-+})^2$$

$$+4\alpha_1(r)\alpha_2(r)\frac{r^2 + y^2}{\Delta_r \hat{\Delta}_r}(2\mathbf{M}r - \mathbf{q}/2)dr^2 - 4\beta_1(y)\beta_2(y)\frac{r^2 + y^2}{\Delta_y \hat{\Delta}_y}(2\mathbf{N}y + \mathbf{q}/2)dy^2,$$

where

$$K = k_\mu dx^\mu\,, \quad N = n_\mu dx^\mu\,, \quad L^{+-} = l_\mu^{+-} dx^\mu\,, \quad L^{-+} = l_\mu^{-+} dx^\mu$$

are AdS_4 Kerr-Schild 1-forms, $\alpha_1 + \alpha_2 = \beta_1 + \beta_2 = 1$.

- **Kerr-Newman case** (rotated and charged black hole)

$$ds^2 = ds_0^2 + \frac{2\mathrm{M}r - \frac{\mathbf{q}}{2}}{r^2 + y^2} k_\mu k_\nu dx^\mu dx^\nu \,.$$

$$C_2 = 1 + \lambda^2 a^2 \,, \qquad C_4 = C_2^2 + 4\lambda^2 a^2 \,.$$

- **Reissner-Nordström** (static charged black hole)

$$C_4 = C_2^2 \neq 0 \,, \quad (K_A{}^C K_C{}^B = C_2 \delta_A{}^B)$$

- **Double Kerr-Schild solution** (complex form of BH)

$$\alpha_1 = \beta_1 = 1, \quad \alpha_2 = \beta_2 = 0$$

$$ds^2 = ds_0^2 + \frac{2\mathrm{M}r - \mathbf{q}/2}{r^2 + y^2} KK - \frac{2\mathrm{N}y + \mathbf{q}/2}{r^2 + y^2} L^{+-} L^{+-}$$

Conclusions

- It is shown that a wide class of black hole metrics (**Carter-Plebanski)** admits simple unfolded description in terms of Killing and source-free Maxwell fields. The system is obtained as a parametric deformation of AdS_4 global symmetry equation. Two deformation parameters $\mathcal{M} \in \mathbb{C}$ and $\mathbf{q} \in \mathbb{R}$ are associated with black hole mass $\mathbf{M}$=Re $\mathcal{M}$, NUT charge $\mathbf{N}$=Im $\mathcal{M}$ and electro-magnetic charges $\mathbf{q} = e^2 + \mathrm{g}^2$. Black hole kinematic characteristics are expressed via two first integrals of the unfolded system

- Type of a black hole whether it is rotated or not is defined by the values of AdS_4 invariants (Casimir operators). In particular, **static black hole** is defined by

$$C_4 = C_2^2$$

- The proposed formulation gives rise to a **coordinate-free description** of the black hole metric in AdS_4

- **Black hole Fronsdal fields** result as a simple consequence in the unfolded system