Matrix Models inspired constructions for Topological Theories

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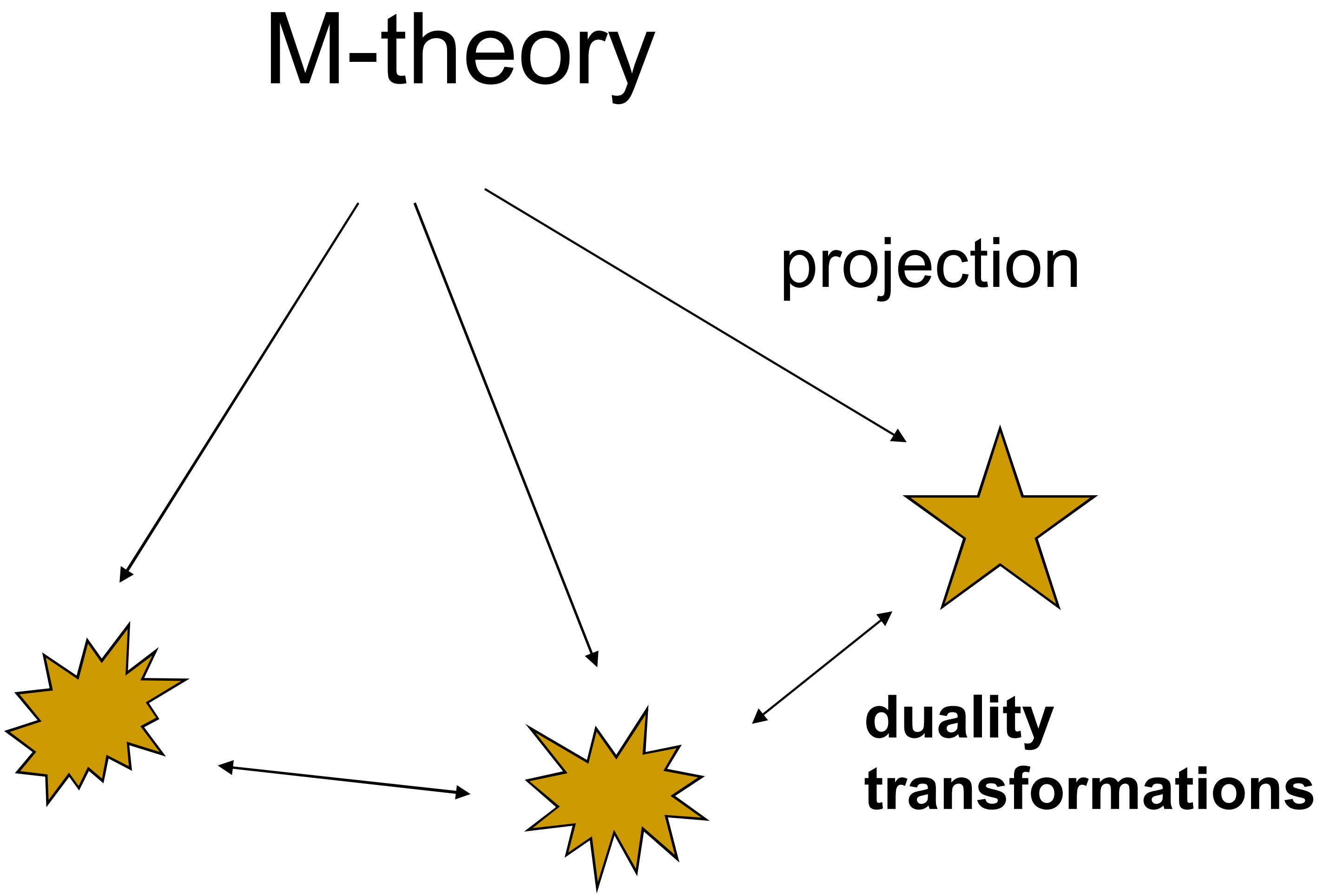

M-theory
projection
duality
transformations

Global Matrix Model

given over Riemann surface

smoothly interpolates between different matrix models

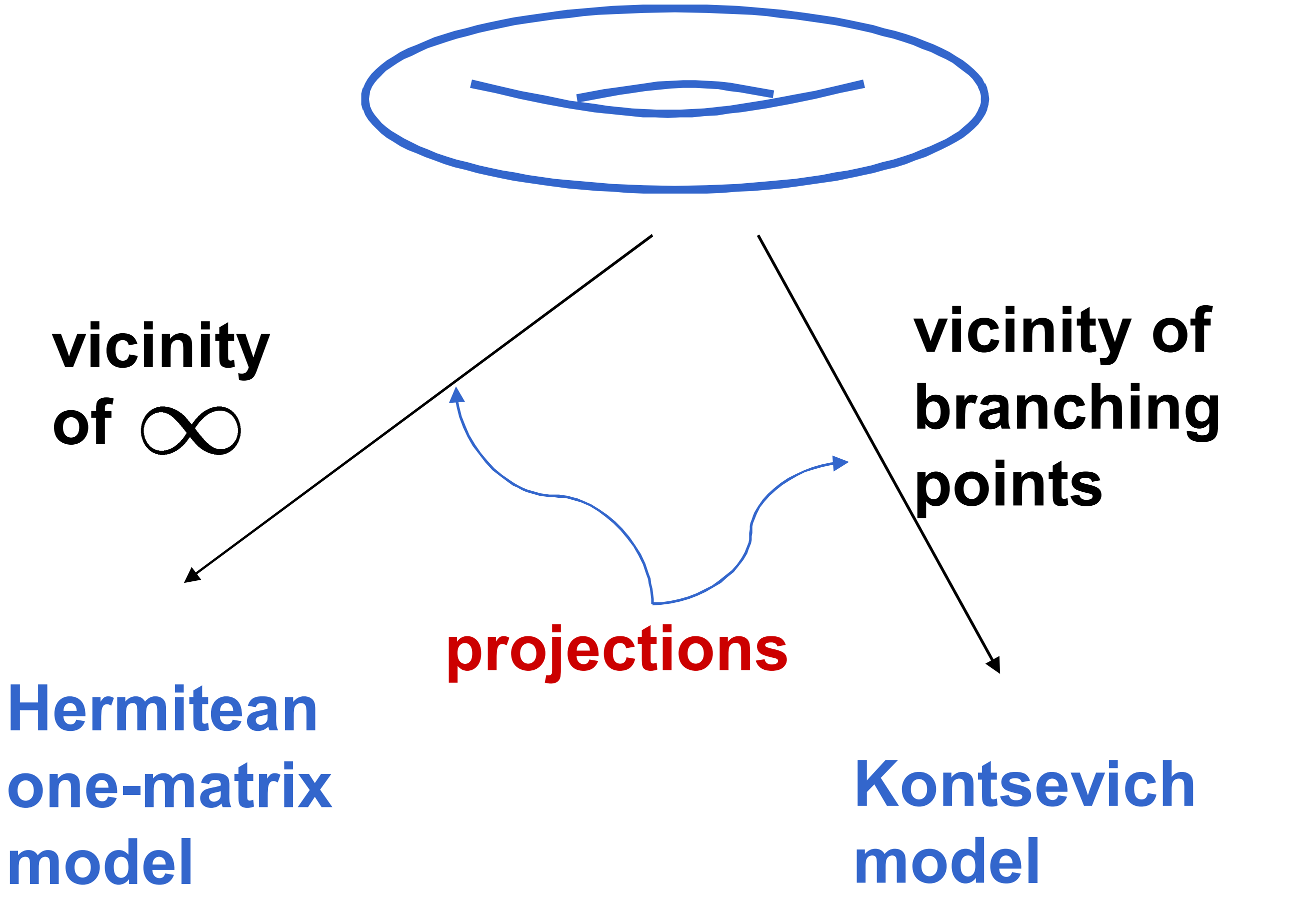
Concrete matrix models

are associated with vicinities of marked points
on the Riemann surface

are obtained from the global matrix model by a projection

Basic example:

Global Matrix Model



The procedure

- Matrix model partition function is defined as a solution to the Virasoro algebra (= loop equations)

- The Virasoro algebra is constructed from the U(1)-currents via the Sugawara construction

- Projection at the vicinity of the marked point is accompanied with a conjugation operator, because of changing the local parameter

- This changing the local parameter induce the corresponding change of time variables

Riemann surface $y_G^2 = (z - a_+)(z - a_-)$

Global U(1)-current:

$$\hat{\mathbf{J}}^o(z, y_G|T, S) \equiv \sum_{k=0}^{\infty} \left\{ \frac{1}{2}\left(k + \frac{1}{2}\right)(T_k + zS_k) y_G^{2k-1} dz + g^2 \frac{dz}{y_G^{2k+3}} \left(\frac{\partial}{\partial \tilde{T}_k} + z \frac{\partial}{\partial \tilde{S}_k} \right) \right\}$$

U(1)-current nearby $z = \infty$

$$\hat{J}_G(z|t) = \sum_{k=0}^{\infty} \left\{ \frac{1}{2} k t_k z^{k-1} dz + g^2 \frac{dz}{z^{k+1}} \frac{\partial}{\partial t_k} \right\}$$

U(1)-current nearby $z = a_\pm$

$$\hat{J}_K(\xi|\tau) = \sum_{k=0}^{\infty} \left\{ \frac{1}{2}(k + \frac{1}{2}) \tau_k \xi^{2k} d\xi + g^2 \frac{d\xi}{\xi^{2k+2}} \frac{\partial}{\partial \tau_k} \right\}$$

Virasoro constraints:

$$T_-(z) \equiv \sum_{n<0} \frac{L_n}{z^{n+2}} = :J^2(z): \qquad T_-(z)Z = 0$$

⇕

Defining equation:

$$\oint_{z,\infty} dz' T_-(z') K(z,z') Z = 0$$

$$K(z,z') \equiv \frac{dz}{dz'} \frac{1}{z-z'} \left[\frac{1}{y(z)} - \frac{1}{y(z')} \right]$$

Duality transformation:

$$Z_G(t) = e^{U_G + \widehat{U}_K} Z_K(\tau_+) Z_K(\tau_-)$$

Partition function of the Hermitean one-matrix model

Conjugation operators (due to the projection)

Partition function of the Kontsevich model

All U have the structure $\sum A_{kl} T_k T_l + B_{kl} T_k \frac{\partial}{\partial T_l} + C_{kl} \frac{\partial^2}{\partial T_k \partial T_l}$

T_k **are the corresponding times**

Changing local parameter leads to a quadratic exponential in times

$$e^{-U_G}\hat{\mathbf{J}}^o(z, y_G|T, S)e^{U_G} = \hat{J}_G(z|t)$$

Matrix model partition functions are τ-functions

Hence, equivalent hierarchies:

Changing local parameter $\mu \to f(\mu) = \mu + \sum_{i=-\infty}^{0} f_i \mu^i$

in the vicinity of $\mu = \infty$ **gives rise to a triangle change**

of times and changes the τ **-function as**

$$\tau(t) = e^{-\frac{1}{2}\sum_{ij} A_{ij}\tilde{t}_i\tilde{t}_j}\tilde{\tau}(\tilde{t}), \qquad A_{ij} = \mathrm{res}_{\mu=\infty} f^i(\mu) d_\mu f_+^i(\mu)$$

Matrix model

↔

Topological Theory

- Double scaling limit of matrix models = 2d gravity
- Unitary matrix model = 2d Yang-Mills theory
- Generalized Kontsevich model = 2d gravity

New examples:

- Hodge integrals with λ-classes = Twisted Kontsevich model
- Hurwitz numbers = new Kontsevich-type matrix model

global matrix models

↔

global topological theories

Ingredients:

- Riemann surface
- Global algebra
- Projection

Technical tool – constraint (Virasoro) algebra.
It completely fixes the theory.

Constraints are equivalent to the defining equations

defined over Riemann surface
with given additional structure –
the Dijkgraaf-Vafa differential

A.Alexandrov, A.Mironov, A.Morozov; B.Eynard

The construction which includes the defining equations is more general than any concrete matrix model or topological theory and gives the global theory.

General defining equation

Consider the curve (Riemann surface) with the involution.

$K(z,z')$ **is actually a ratio of the Green function on the Riemann surface (which is the primitive of the Bergmann kernel w.r.t. the second argument calculated from** z' **to** $\tilde{z}'$**) and the Dijkraaf-Vafa differential:**

$$K(z,z') = \frac{< \partial\phi(z)\,\phi(z') > - < \partial\phi(z)\,\phi(\tilde{z}') >}{\Omega_{DV}(z') - \Omega_{DV}(\tilde{z}')}$$

Tilde relates two points connected by the involution

New construction: Hurwitz numbers vs. Hodge integrals

1) They are related via the ELSV formula
(Ekedahl, Lando, Shapiro, Vainshtein)

2) The generating functions are related
by the construction above

$$F = \sum_{q,p} u^{2q} g^{2p} (-)^q \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_i} \delta \left(\sum_{i=1}^{n} (k_i - 1) - (3p - 3 - q) \right) I_q^{(p)}(k_i)\, T_{k_1} \dots T_{k_n}$$

$$I_q^{(p)}(k_1, \dots, k_n) = \int_{\mathcal{M}_{p,n}} \lambda_q \psi_1^{k_1} \dots \psi_n^{k_n}$$

Hurwitz numbers

$$H = \frac{1}{g^2}\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{p;m_i;M}\delta\left(\sum_{i=1}^{n}(m_i+1)+2p-2-M\right)\frac{u^{3M}g^{2p}}{M!}h(p\,|m_i)p_{m_1}\dots p_{m_n}$$

e^H **is a** τ **-function of** $t_k = p_k/k$

Change of local parameter: $f(\mu) = u^3(\mu+1)e^{-\frac{1}{1+\mu}}$

Equivalent hierarchy: $e^{F(q)} = e^{H_2(p)}e^{H(p)}$

$H_2(p)$ **is quadratic in times,**
q are new time variables

$$T_0 = u^4 q_1$$

$$T_1 = u^3 \oint w(1+w)^2 dp = u^6 q_1 + 2u^9 q_2 + u^{12} q_3$$

$$T_2 = u^8 q_1 + 6u^{11} q_2 + 12u^{14} q_3 + 10u^{17} q_4 + 3u^{20} q_5$$

...

$e^{F(q)}$ **is a** τ **-function**

$e^{F(T)}$ **is not a** τ **-function**

$$T_k = u^{2k+1} \sum_{n=1}^{\infty} \frac{n^{n+k}}{n!} u^{3n} p_n$$

Using the defining equation for the Lambert curve $x = (z+1)e^{-z}$

$$e^{F(T)} = e^{\hat{U}} e^{H(p)}$$

$$\hat{U} = \exp\left(\sum_k s_k u^{4k+2} \hat{M}_{2k+1}\right)$$

$$\hat{M}_{2k+1} \equiv \sum_l \hat{T}_l \frac{\partial}{\partial T_{l+k}} - \frac{1}{2} \sum_{a+b=2k} (-)^a \frac{\partial^2}{\partial T_a \partial T_b}$$

$$s_k = \frac{B_{k+1}}{k(k+1)}$$

Conclusion:

One can unify an array of various topological theories into a global topological theory which projections, concrete topological theories are related by duality transformations.

Sometimes these theories can be realized via matrix models.

Technical tool for this scheme is realized in the defining equation.

To construct the global theory in topological terms and to reveal the integrable properties of the construction still remain open problems.