Supergravity Black Holes and Billiards and Liouville integrable structure of dual Borel algebras

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Supergravity solutions

1) space-like p-branes as the cosmic billiards, or

2) time-like p-branes as several rotational invariant black-holes in $D = 4$ and more general solitonic branes in diverse dimensions

reduce to geodesic equations on coset manifolds of the type

$$\mathcal{M} = \frac{\mathrm{U}}{\mathrm{H}} \quad or \quad \mathcal{M}^\star = \frac{\mathrm{U}}{\mathrm{H}^\star} \simeq \exp\left[\mathrm{Solv}_{\mathcal{M}}\right]$$

SL(p+q)/SO(p, q) cosets

$$\eta = \mathrm{diag}(\underbrace{-1, ..., -1}_{p}, \underbrace{+1, ... + 1}_{q})$$

$$L = L_{>} + \eta\, L_{>}^{T}\, \eta$$

$$(L\ \eta)^{T} = L\ \eta$$

$$W = L_{>} - \eta\, L_{>}^{T}\, \eta \in so(p, q)$$

$$(W\ \eta)^{T} = -W\ \eta$$

$$L + W \equiv \mathcal{L}^A \, T_A \in \, Solv(sl(p+q))$$

$$[T_A, \; T_C] = f_{AC}{}^B \, T_B$$

$$\{T_A, \; A = 1, ..., \frac{1}{2}(p+q)(p+q+1)\}$$

$$[\mathbb{H},\mathbb{H}]\subset\mathbb{H},\quad [\mathbb{H},\mathbb{K}]\subset\mathbb{K},\quad [\mathbb{K},\mathbb{K}]\subset\mathbb{H}$$

$$\mathbb{H}\subset\mathbb{U},\quad \mathbb{U}=\mathbb{H}\oplus\mathbb{K}$$

- $\mathbb{L}(\phi) = \prod_{I=m}^{I=1} \exp\left[\varphi_I\, E^{\alpha_I}\right] \exp\left[h_i \mathcal{H}^i\right]$

SL(p+q)/SO(p,q) solvable coset representative

- $L(t) = \sum_i \mathrm{Tr}\left(\mathbb{L}^{-1}\frac{d}{dt}\mathbb{L}\,\mathsf{K}_i\right)\mathsf{K}_i\ ,$
- $W(t) = \sum_\ell \mathrm{Tr}\left(\mathbb{L}^{-1}\frac{d}{dt}\mathbb{L}\,\mathsf{H}_\ell\right)\mathsf{H}_\ell$

$$\frac{d}{dt}L = [W\,,\ L]$$

$$W = \Pi(L): = L_> - L_<$$

$$\frac{d}{dt}\mathcal{L}_A + f_{AC}{}^{B}\,\mathcal{L}_B\,\frac{\partial \mathcal{H}}{\partial \mathcal{L}_C} = 0$$

$$\mathcal{H} = tr(L^2) \equiv \frac{1}{2}\,\mathcal{L}_B\,\mathcal{L}^B$$

$$\mathcal{L}_B \equiv g_{BC}\mathcal{L}^C$$

Lie-Poisson structure

$$\{\phi_1,\ \phi_2\} = f_{AC}{}^{B}\,\mathcal{L}_B \frac{\partial \phi_1}{\partial \mathcal{L}_A}\frac{\partial \phi_2}{\partial \mathcal{L}_C}$$

$$\frac{d}{d\tau}\mathcal{L}_A + \{\mathcal{L}_A,\ \mathcal{H}\} = 0$$

Liouville integrability

$$\det\{(L-\mu)_{ij}:\ \alpha+1\leq i\leq p+q,\ 1\leq j\leq p+q-2\alpha\}$$

$$=\mathcal{E}_{\alpha 0}\left(\mu^{p+q-2\alpha}+\sum_{\beta=1}^{p+q-2\alpha}\mathcal{H}_{\alpha\beta}\,\mu^{p+q-2\alpha-\beta}\right)$$

$$\alpha=0,\ldots,\left[\frac{p+q}{2}\right]$$

$$\{\mathcal{H}_{\alpha\beta},\ \mathcal{H}_{\delta\gamma}\}=0 \qquad \{\mathcal{C}_l,\ \mathcal{L}_A\}=0$$

N	d_N	p_N	c_N	$2m$
2	3	2	1	2
3	6	4	2	4
4	10	6	2	8
5	15	9	3	12
6	21	12	3	18
7	28	16	4	24
8	36	20	4	32
9	45	25	5	40
10	55	30	5	50
11	66	36	6	60
12	78	42	6	72
13	91	49	7	84
14	105	56	7	98
15	120	64	8	112
16	136	72	8	128

$$\mathfrak{gl}(\mathrm{N}) \supset \widehat{\mathbb{B}}_{\mathrm{N}} \ni \begin{pmatrix} \star & \dots & \dots & \dots & \dots & \star \\ 0 & \star & \dots & \dots & \dots & \star \\ \vdots & 0 & \star & \dots & \dots & \star \\ \vdots & \vdots & 0 & \star & \dots & \star \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \star \end{pmatrix}$$

$$\dim \widehat{\mathbb{B}}_{\mathrm{N}} \equiv d_{\mathrm{N}} = \frac{\mathrm{N}(\mathrm{N}+1)}{2}$$

	$d = \dim \widehat{\mathbb{B}}$	$p \equiv \#$ funct. in inv.	$c \equiv \#$ of Casim.	$2m = $ dim of orbits
$\widehat{\mathbb{B}}_{2\nu}$	$\nu(2\nu+1)$	$\nu^2+\nu$	ν	$2\nu^2$
$\widehat{\mathbb{B}}_{2\nu+1}$	$(\nu+1)(2\nu+1)$	$\nu^2+2\nu+1$	$\nu+1$	$2(\nu^2+\nu)$

General solution

$$L(t) \;=\; \mathcal{Q}(\mathcal{C})\, L_0\, \left(\mathcal{Q}(\mathcal{C})\right)^{-1}$$

$$\mathcal{Q}_{ij}(\mathcal{C}) = \frac{1}{\sqrt{\mathfrak{D}_i(t)\mathfrak{D}_{i-1}(t)}}\,\mathrm{Det}\begin{pmatrix} \mathcal{C}_{1,1}(t) & \ldots & \mathcal{C}_{1,i-1}(t) & (\mathcal{C}^{\frac{1}{2}}(t))_{1,j} \\ \vdots & \vdots & \vdots & \vdots \\ \mathcal{C}_{i,1}(t) & \ldots & \mathcal{C}_{i,i-1}(t) & (\mathcal{C}^{\frac{1}{2}}(t))_{i,j} \end{pmatrix}$$

$$\mathcal{C}(t) \;=\; e^{-2\,t\,L_0}$$

$$\mathfrak{D}_i(t) \;=\; \mathrm{Det}\begin{pmatrix} \mathcal{C}_{1,1}(t) & \ldots & \mathcal{C}_{1,i}(t) \\ \vdots & \vdots & \vdots \\ \mathcal{C}_{i,1}(t) & \ldots & \mathcal{C}_{i,i}(t) \end{pmatrix}$$

Triangular embedding in SL(N)/SO(p,N-p) and integrability of the Lorentzian cosets U/H*

Statement 3.1 << *Let* N *be the real dimension of the fundamental representation of* $\mathbb{U}$. *For each choice of* $\mathbb{H}$ *or* $\mathbb{H}^\star$ *there exist a suitable integer* $p \leq [\frac{N}{2}]$ *and a diagonal metric*

$$\eta = \mathrm{diag}(\underbrace{-1, \ldots, -1}_{p}, \underbrace{+1, \ldots +1}_{q \equiv N-p}) \ ,$$

such that we have a canonical embedding

$$\begin{aligned} \mathbb{U} \ &\hookrightarrow \ \mathfrak{sl}(\mathrm{N}) \ , \\ \mathbb{U} \supset \mathbb{H}^\star \ &\hookrightarrow \ \mathfrak{so}(\mathrm{p}, \mathrm{N}-\mathrm{p}) \subset \mathfrak{sl}(\mathrm{N}) \ . \end{aligned}$$

This embedding is determined by the choice of the basis where Solv $(\mathrm{U}/\mathrm{H}^\star)$ *is made by upper triangular matrices. In the same basis the elements of* $\mathbb{K}$ *are* η*-symmetric matrices while those of* $\mathbb{H}^\star$ *are* η*-antisymmetric ones, namely:*

$$\begin{aligned} \forall \, \mathrm{K} \in \mathbb{K} \ : \quad \eta \, \mathrm{K}^T &= \mathrm{K}^T \eta \ , \\ \forall \, \mathrm{H} \in \mathbb{H}^\star \ : \quad \eta \, \mathrm{H}^T &= -\mathrm{H}^T \eta \ . \end{aligned}$$

>>

$$\frac{d}{dt} L = [W \, , \, L]$$

The paradigmatic example: SL(3)/SO(1,2) coset

$$L(t) = \begin{pmatrix} \frac{1}{\sqrt{2}}Y_1(t) - \frac{1}{\sqrt{6}}Y_2(t) & -\frac{1}{2}Y_3(t) & -\frac{1}{2}Y_5(t) \\ \frac{1}{2}Y_3(t) & -\frac{1}{\sqrt{2}}Y_1(t) - \frac{1}{\sqrt{6}}Y_2(t) & -\frac{1}{2}Y_4(t) \\ \frac{1}{2}Y_5(t) & -\frac{1}{2}Y_4(t) & \sqrt{\frac{2}{3}}Y_2(t) \end{pmatrix}$$

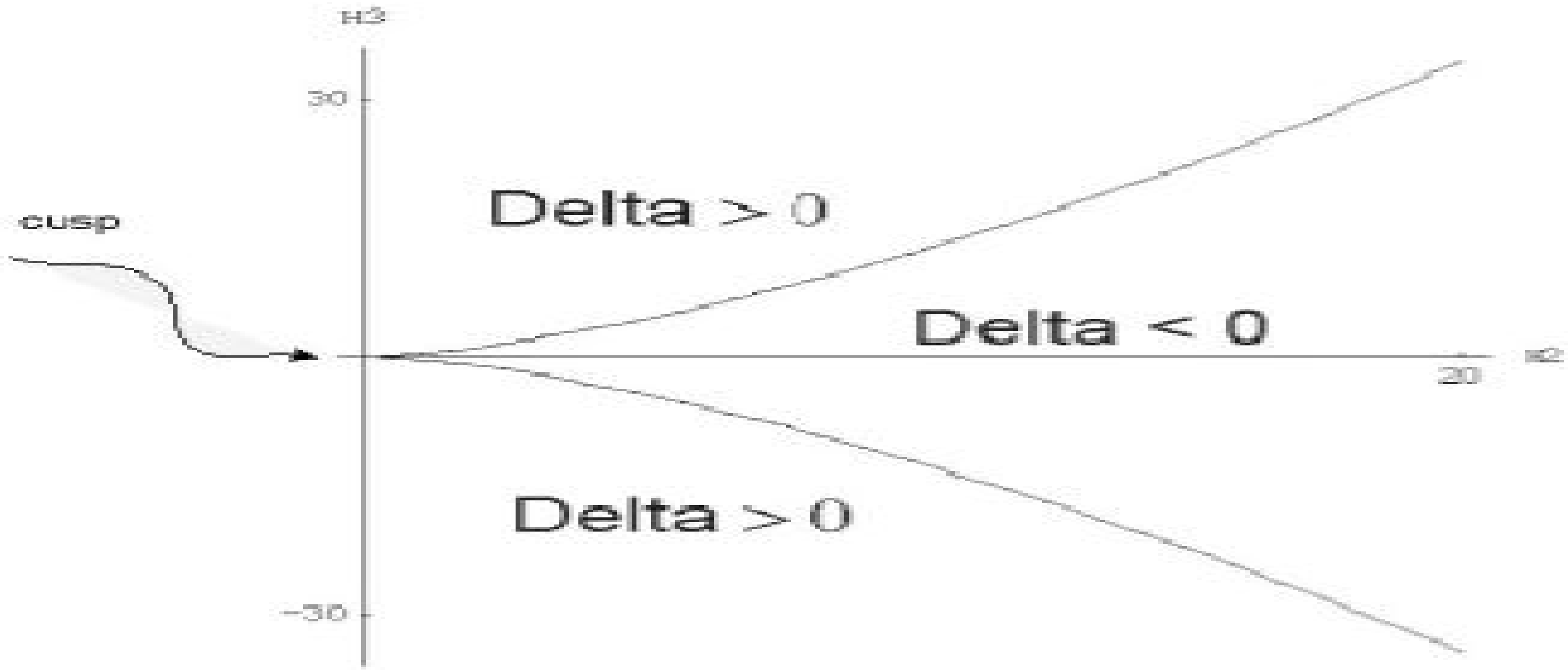
$$\begin{aligned}
-\frac{Y_3(t)^2}{\sqrt{2}} - \frac{Y_4(t)^2}{2\sqrt{2}} - \frac{Y_5(t)^2}{2\sqrt{2}} + \frac{d}{dt}Y_1(t) &= 0 , \\
-\frac{1}{2}\sqrt{\frac{3}{2}}Y_4(t)^2 + \frac{1}{2}\sqrt{\frac{3}{2}}Y_5(t)^2 + \frac{d}{dt}Y_2(t) &= 0 , \\
-\sqrt{2}Y_1(t)Y_3(t) - Y_4(t)Y_5(t) + \frac{d}{dt}Y_3(t) &= 0 , \\
\frac{Y_1(t)Y_4(t)}{\sqrt{2}} + \sqrt{\frac{3}{2}}Y_2(t)Y_4(t) - Y_3(t)Y_5(t) + \frac{d}{dt}Y_4(t) &= 0 , \\
-\frac{Y_1(t)Y_5(t)}{\sqrt{2}} + \sqrt{\frac{3}{2}}Y_2(t)Y_5(t) + \frac{d}{dt}Y_5(t) &= 0 ,
\end{aligned}$$

$$\mathfrak{h}_2 \doteq \mathfrak{h}_{02} = Y_1(t)^2 + Y_2(t)^2 - \frac{1}{2}Y_3(t)^2 + \frac{1}{2}Y_4(t)^2 - \frac{1}{2}Y_5(t)^2 ,$$

$$\mathfrak{h}_3 \doteq \mathfrak{h}_{03} = \frac{Y_2(t)^3}{3\sqrt{6}} - \frac{Y_1(t)^2 Y_2(t)}{\sqrt{6}} + \frac{Y_3(t)^2 Y_2(t)}{2\sqrt{6}} + \frac{Y_4(t)^2 Y_2(t)}{4\sqrt{6}} - \frac{Y_5(t)^2 Y_2(t)}{4\sqrt{6}} - \frac{Y_1(t) Y_4(t)^2}{4\sqrt{2}} - \frac{Y_1(t) Y_5(t)^2}{4\sqrt{2}} + \frac{1}{4} Y_3(t) Y_4(t) Y_5(t)$$

$$\mathcal{C}_1 \doteq \mathfrak{h}_{01} = 0$$

$$\mathcal{C}_2 \doteq \mathfrak{h}_{11} = \frac{Y_1(t)}{\sqrt{2}} + \frac{Y_2(t)}{\sqrt{6}} - \frac{Y_3(t) Y_4(t)}{2Y_5(t)}$$



$$\Delta \equiv -12\mathfrak{h}_2^3 + 81\,\mathfrak{h}_3^2$$

- In the region where $\Delta < 0$ we have three distinct real eigenvalues.
- In the region where $\Delta > 0$ there is one real eigenvalue and a pair of complex conjugate eigenvalues.
- The locus $\Delta = 0$ corresponds to orbits admitting an enhanced symmetry, except at the cusp.

Intrinsic characterization of the Nilpotent orbits: vanishing of polynomial hamiltonians.

Conjecture:

Cuspidal orbits of nilpotent Lax operators can also be found by searching for eigenstates of the noncompact generators of H*.

Null eigenstates give orbits with enhanced symmetry (stability subgroup).

Eigenstates of non-vanishing eigenvalue occur only at the cusp.

$$L \in \mathcal{C}_0 \Leftrightarrow \forall G \in \mathfrak{so}(1,2) \; : \; [G\, , \, L] = \mu\, L$$

$$L_1(t) = \begin{pmatrix} \frac{2t}{1-2t^2} & -\frac{\sqrt{1+2t^2}}{1-2t^2} & 0 \\ \frac{\sqrt{1+2t^2}}{1-2t^2} & \frac{4t}{4t^4-1} & \frac{\sqrt{1-2t^2}}{1+2t^2} \\ 0 & \frac{\sqrt{1-2t^2}}{1+2t^2} & \frac{2t}{1+2t^2} \end{pmatrix} \qquad (L_1)^3 = 0 \qquad \mu = 1$$

$$L_2(t) = \frac{1}{1+2t} \begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad (L_2)^2 = 0 \qquad \mu = 2$$

Conclusions and Outlook

1. The integrability of all the various symmetric-solvable cosets for generic coajoint group orbits, both Euclidian and Lorentzian, follows from the Liouville integrability of SL(p+q)/SO(p, q) cosets based on Borel subalgebra $B_{(p+q)}$ of sl(p+q).
2. The norm on any solvable Lie algebra S is not an independent external datum, rather it is intrinsically defined by the restriction to S of the unique quadratic hamiltonian on $B_{(p+q)}$, once the embedding S → $B_{(p+q)}$ has been defined.
3. All symmetric cosets U/H* have integrable geodesic equations if the Lie algebra of U is non-compact and the Lie algebra of H* is any of the real sections contained in U of the complexication H_C of H ⊂ U, the former being the maximal compact subalgebra of the latter.
4. The explicit integration algorithm has a universal form.

Open questions

1. Liouville integrability of singular group orbits?
2. The relation of the Hamiltonians with the physical invariants of the solution, like the entropy or the total mass?
3. The solvable parametrization covers only open branches of the space and the question is how to glue together different branches (global topology of the solution space)?

Thank you for attention!

Parameters of the time flows

From initial data we obtain the time flow (complete integral)

$$\mathcal{I}_K \quad : \quad L_0 \mapsto L(t, L_0)$$

Initial data are specified by a pair: an element of the non-compact Cartan Subalgebra and an element of maximal compact group:

$$\mathcal{C}_0 \in \mathsf{CSA} \bigcap \mathbb{K} \quad ; \quad \mathcal{O} \in \mathsf{H} \; .$$

$$L_0 = \mathcal{O}^T \mathcal{C}_0 \mathcal{O}$$

Properties of the flows

The flow is isospectral

$$\forall\, t \in \mathbb{R} \quad : \quad \text{Eigenvalues}\,[L(t)] =$$
$$= \{\lambda_1 \ldots \lambda_N\} = \text{const}$$

$$\boxed{L(t) = \mathcal{O}^T(t)\, \mathcal{C}_0\, \mathcal{O}(t)}$$

The asymptotic values of the Lax operator are diagonal (Kasner epochs)

$$\lim_{t \to \pm\infty} L(t) = L_{\pm\infty} \in \text{CSA}$$

$$\lim_{t \to \pm\infty} \mathcal{O}(t) \in \text{Weyl}\,(\mathbb{U}_{\text{TS}})$$

Main Points

Definition

<< A supergravity billiard is a one-dimensional σ-model whose target space is a non-compact coset manifold U/H*, metrically equivalent, in force of a general theorem, to a solvable group manifold* $\exp\left[Solv\left(\mathrm{U}/\mathrm{H}\right)\right]$. *>>*

Because ***t-dependent*** supergravity field equations are equivalent to the ***geodesic equations*** for a manifold

U/H

Statement

Supergravity billiards are exactly integrable by means of a general algorithm constructing the Toda-like flow

Because **U/H** is always metrically equivalent to a solvable group manifold **exp[Solv(U/H)]** and this defines a canonical embedding

$$\mathbb{U} \hookrightarrow \mathfrak{sl}(N,\mathbb{R}) \;,$$
$$\mathbb{U} \supset \mathbb{H} \hookrightarrow \mathfrak{so}(N) \subset \mathfrak{sl}(N,\mathbb{R}) \;.$$

The discovered Principle

<< The asymptotic states at $t = \pm\infty$ are in one-to-one correspondence with the elements $w_i \in \mathrm{Weyl}(\mathbb{U})$. The Weyl group admits a natural ordering in terms of $\ell_T(w)$, i.e. the number of transpositions of the corresponding permutation when $\mathrm{Weyl}(\mathrm{U})$ is embedded in the symmetric group. Time flows goes always in the direction of increasing ℓ_T which, therefore, plays the role of entropy. >>

The relevant **Weyl group** is that of the **Tits Satake** projection. *It is a property of a* ***universality class*** *of theories.*

$$\Pi_{\mathrm{TS}} \quad : \quad \frac{\mathrm{U}}{\mathrm{H}} \longrightarrow \frac{\mathrm{U}_{\mathrm{TS}}}{\mathrm{H}_{\mathrm{TS}}}$$

There is an interesting topology of parameter space for the LAX EQUATION

Parameter space	$\mathcal{P} = \dfrac{\mathsf{H}}{\mathsf{G}_{\mathsf{paint}}} / \mathsf{Weyl}(\mathbb{U})$
Proposition	*<<Consider now the $2^{\mathrm{N}} - 1$ minors of $\mathcal{O}(t)$ obtained by intersecting the first k columns with any set of k-rows, for $k = 1, \ldots, \mathrm{N} - 1$. If any of these minors vanishes at any finite time $t \neq \pm\infty$ then it is constant and vanishes at all times.>>*
Trapped submanifolds	There are $N^2 - 1$ **trapped hypersurfaces** $\Sigma_i \subset \mathcal{P}$. defined by the vanishing of one of the minors. They can be intersected.
ARROW OF TIME	*On any (trapped) manifold the flow is from the lowest, to the highest accessible Weyl element*

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Standard FRW cosmology is concerned with studying the evolution of specific general relativity solutions, but we want to ask what more general type of evolution is conceivable just under GR rules.

What if we abandon isotropy?

The Kasner universe: an empty, homogeneous, but non-isotropic universe

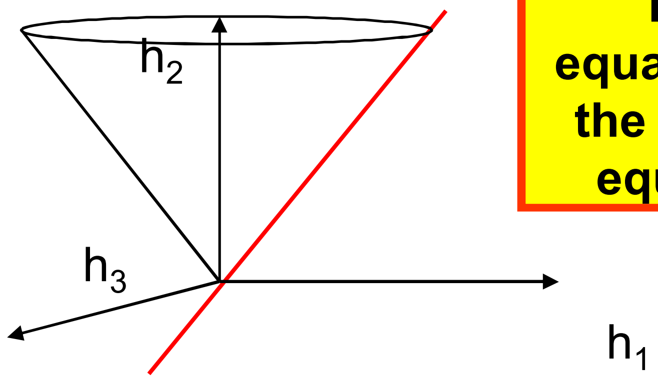
$$g_{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & a_1^2(t) & & \\ & & a_2^2(t) & \\ 0 & & & a_3^2(t) \end{pmatrix} \Rightarrow \quad a_i(t) = t^{p_i}, \quad \sum_{i=1}^{3} p_i^2 = \sum_{i=1}^{3} p_i = 1$$

These equations are the Einstein equations

Some of the scale factors expand, but some other have to contract: an anisotropic universe is not static even in the absence of matter!

Useful pictorial representation:
A light-like trajectory of a ball in the lorentzian space of

$h_i(t) = \log[a_i(t)]$



Introducing Billiard Walls

Let us now consider, the coupling of a vector field to diagonal gravity $g_{ii} = e^{h_i}$

$$\sqrt{g} F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} = \sum_{i,j} e^{-h_i - h_j + \frac{1}{2}\sum h_k} (F_{ij})^2$$

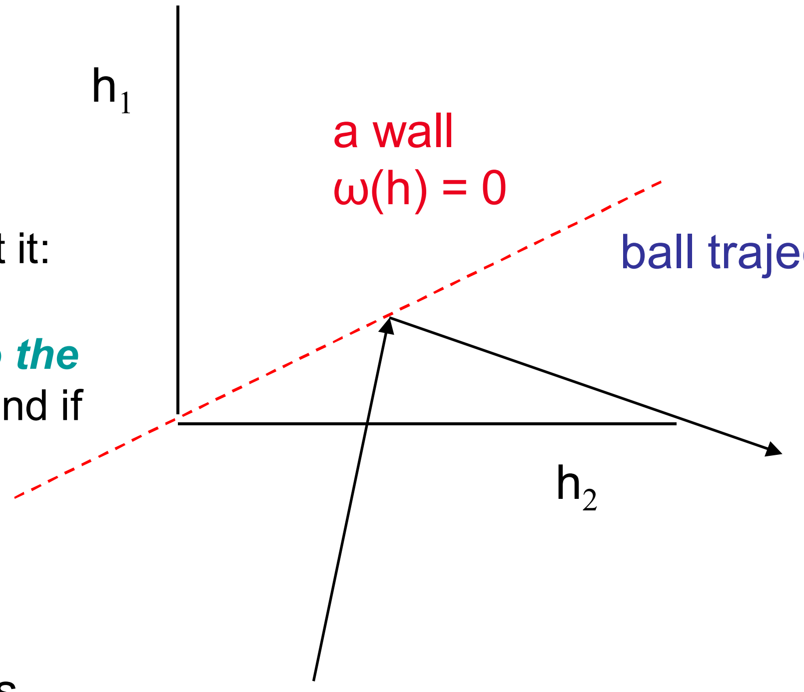
If ***F_{ij} = const*** this term adds a ***potential*** to the ball's hamiltonian

$$V = \sum_{ij} e^{\omega_{ij}(h)} F_{ij}^2$$

Asymptoticaly $|h| \to \infty$

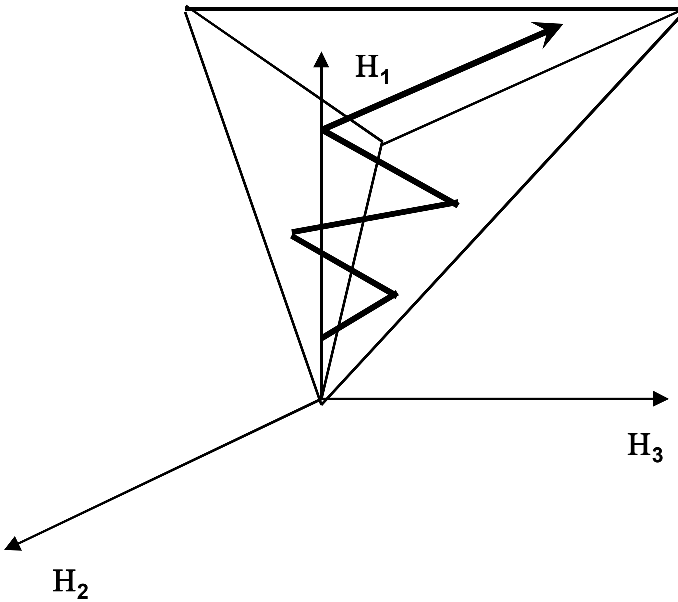
- $\omega_{ij}(h) < 0$ — *Free motion (Kasner Epoch)*
- $\omega_{ij}(h) > 0$ — *Inaccessible region*
- $\omega_{ij}(h) = 0$ — *Wall position or bounce condition*

The Rigid billiard



When the ball reaches the wall it ***bounces*** against it: geometric reflection.
It means that the ***space directions transverse to the wall change their behaviour***: they begin to expand if they were contracting and vice versa.

Billiard table: the configuration of the walls

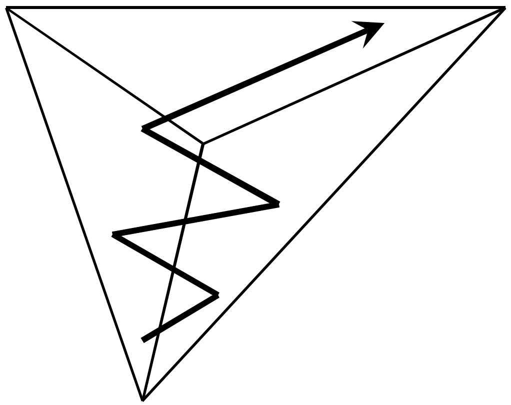


-- the full evolution of such a universe is a sequence of Kasner epochs with bounces between them
-- the number of large (visible) dimensions can vary in time dynamically
-- the number of bounces and the positions of the walls depend on the field content of the theory: microscopical input

Smooth Billiards and dualities

Asymptotically any time—dependent solution defines a ***zigzag*** in **ln a$_i$** space

The Supergravity billiard is completely determined by U-duality group



Damour, Henneaux, Nicolai 2002 --

h-space	⟺	***CSA of the U algebra***
walls	⟺	***hyperplanes orthogonal to positive roots*** $\alpha(h_i)$
bounces	⟺	***Weyl reflections***
billiard region	⟺	***Weyl chamber***

Smooth billiards: *Exact cosmological solutions can be constructed using U-duality (**in fact billiards are exactly integrable**)*

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series of papers

bounces	⟺	***Smooth Weyl reflections***
walls	⟺	***Dynamical hyperplanes***

The space-like p-brane solutions that have an Euclidian world-volume and are time-dependent, all fields being functions of the time parameter t.

The time-like p-brane solutions that have a Minkowskian world volume and are stationary, the fields depending on another parameter t, typically measuring the distance from the brane.