The embedding-tensor formalism with fields and antifields.

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Intro and content: 2 formalisms

- There are basic supergravities and deformations.
- The basic ones are known.
  The embedding tensor formalism (Nicolai, Samtleben, de Wit, Trigiante) gives a possible way to describe the deformations.
- This formalism has many features (soft algebra, reducible gauge transformations, on-shell algebra) for which the field-antifield formalism (Batalin-Vilkovisky) is the appropriate language.

The map of supergravities.

Dimensions and # of supersymmetries

D		32		24	20	16		12	8	4
11		M								
10		IIA	IIB			I				
9		N=2				N=1				
8		N=2				N=1				
7		N=4				N=2				
6		(2,2)		(2,1)		(1,1)	(2,0)		(1,0)	
5		N=8		N=6		N=4			N=2	
4		N=8		N=6	N=5	N=4		N=3	N=2	N=1

This classifies the basic supergravities

Basic supergravities and deformations

- Basic supergravities:
  have only gauged supersymmetry and general coordinate transformations (and U(1)'s of vector fields).
  - No potential for the scalars.
  - Only Minkowski vacua.
- In any entry of the table there are 'deformations':
  without changing the kinetic terms of the fields, the couplings are changed.
  - Many deformations are 'gauged supergravities':
    gauging of a YM group, introducing a potential.
  - Produced by fluxes on branes

PS: gauging supersymmetry leads to 'supergravity'.
'gauged supergravity': other symmetries gauged

Gaugings

- Start from all the global symmetries δ_α
- Say which ones are gauged by which gauge fields $A_\mu{}^M$
- This is encoded in an 'embedding tensor' $\Theta_M{}^\alpha$
- The deformations can be obtained by embedding tensors satisfying some constraints.

- For D=4 the 'global symmetries' include as well isometries of the scalar manifold as electric-magnetic duality symmetries, which are **symplectic transformations**

Nicolai, Samtleben, 2000; de Wit, Samtleben and Trigiante, 2005

Vector field strengths are in *2m* – symplectic vectors

symplectic transformations

$$\mathcal{L}_1 = -\tfrac{1}{4}(\mathrm{Re}\, f_{AB}) F^A_{\mu\nu} F^{\mu\nu\, B} + \tfrac{1}{8}(\mathrm{Im}\, f_{AB}) \varepsilon^{\mu\nu\rho\sigma} F^A_{\mu\nu} F^B_{\rho\sigma}$$

→ coupling constants or functions of scalars

$$\begin{array}{llll} \partial^\mu\, \mathrm{Im}\, F^{A-}_{\mu\nu} & = & 0 & \text{Bianchi identities} \\ \partial_\mu\, \mathrm{Im}\, G^{\mu\nu-}_A & = & 0 & \text{Equations of motion.} \end{array}$$

$$F^{\pm A}_{\mu\nu} \equiv \tfrac{1}{2}\left(F^A_{\mu\nu} \pm \tfrac{1}{2}\mathrm{i}\varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma\, A}\right)$$

$$G^{\mu\nu\, -}_A \equiv -2\mathrm{i}\frac{\delta \mathcal{L}_1(F^+, F^-)}{\delta F^{-A}_{\mu\nu}} = \mathrm{i} f_{AB} F^{\mu\nu\, -\, B}$$

Invariance under Gl(*2m*,R)

$$\begin{pmatrix} F'^- \\ G'^- \end{pmatrix} = \mathcal{S} \begin{pmatrix} F^- \\ G^- \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} F^- \\ G^- \end{pmatrix}$$

For consistency:

$$G'^- = (C + \mathrm{i}Df)F^- = (C + \mathrm{i}Df)(A + \mathrm{i}Bf)^{-1} F'^-$$

$$\to \boxed{\mathrm{i}f' = (C + \mathrm{i}Df)(A + \mathrm{i}Bf)^{-1}}$$

should be symmetric

⇒

$$\mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2m, \mathbb{R})$$

Why are not we happy yet ?

- Symplectic symmetry broken by gauging:
  in covariant derivatives appears $A_\mu{}^A$:
  gauge vectors in the ‘upper part’ of the symplectic vectors
- The symplectic symmetry is only valid for Abelian gauge vectors.
- We would like that the embedding of the gauge group in the symplectic group can be done in a symplectic-covariant way:
  any symplectic basis should be possible

Symplectic formalism

- Electric and magnetic gauge fields in symplectic vectors: electric and magnetic components: $A_\mu{}^M = \begin{pmatrix} A_\mu{}^A \\ A_{\mu A} \end{pmatrix}$
- The gauge group is a subgroup of the isometry group G, defined by an embedding tensor. $\left(\partial_\mu - A_\mu{}^M \Theta_M{}^\alpha \delta_\alpha\right) \phi$

all the rigid symmetries

determines which symmetries are gauged, and how:
e.g. also the coupling constants.
There are several constraints on the tensor.

- if usual electric gauging: $\Theta_M{}^\alpha = \begin{pmatrix} \Theta_A{}^\alpha \\ \Theta^{A\alpha} \end{pmatrix} = \begin{pmatrix} \Theta_A{}^\alpha \\ 0 \end{pmatrix}$

Embedding in symplectic matrices

- The main object defines the embedding of the symmetry group in the symplectic group by *2m£ 2m* matrices t_α

$$X_{MN}{}^{P} \equiv \Theta_M{}^{\alpha}(t_\alpha)_N{}^{P}$$

- To be symplectic: $X_{M[NP]} = X_{M[N}{}^{Q}\Omega_{P]Q} = 0$

$$\Omega = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$

- Transformations of electric-magnetic gauge vectors:

$$\delta A_\mu{}^{M} = \partial_\mu \Lambda^M + X_{PQ}{}^{M} A_\mu{}^{P} \Lambda^Q$$

nearly what we expect; however, $X_{PQ}{}^{M} \neq -X_{QP}{}^{M}$

Constraints

remember:

$$X_{MN}{}^{P} \equiv \Theta_M{}^{\alpha}(t_\alpha)_N{}^{P} \qquad X_{MNP} = X_{MN}{}^{Q}\Omega_{PQ} = X_{MPN}$$

1. Locality: $\Theta_M{}^{\alpha}\Omega^{MN}\Theta_N{}^{\beta} = 2\Theta^{A[\alpha}\Theta_A{}^{\beta]} = 0$
2. Closure of gauge algebra

$$f_{\beta\gamma}{}^{\alpha}\Theta_M{}^{\beta}\Theta_N{}^{\gamma} + X_{MN}{}^{P}\Theta_P{}^{\alpha} = 0$$

3. Anomaly cancellation:

$$X_{(MNP)} = \Theta_M{}^{\alpha}\Theta_N{}^{\beta}\Theta_N{}^{\gamma}d_{\alpha\beta\gamma}$$

anomaly tensor from possible chiral fermions

$$\mathcal{A} = d_{\alpha\beta\gamma}\Lambda^{\alpha}F^{\beta}_{\mu\nu}F^{\gamma}_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}$$

no chiral anomalies for *N* ¸ *2* :

then the constraint is the vanishing of symmetric part : $X_{(MNP)}=0$

J. De Rydt, T. Schmidt, M. Trigiante, AVP and M. Zagermann, 2008

Main features of the symplectic formalism

- one needs antisymmetric tensors $B_{\mu\nu\,\alpha}$ to compensate for the extra gauge vectors, with gauge transformations

$$\delta B_{\mu\nu\alpha} = \partial_{[\mu}\Xi_{\nu]\alpha} + \ldots$$

- transformation gauge fields:

$$\delta A_\mu{}^M = \partial_\mu \Lambda^M + X_{PQ}{}^M A_\mu{}^P \Lambda^Q - \tfrac{1}{2}\Omega^{MN}\Theta_N{}^\alpha \Xi_{\mu\,\alpha}$$

- action:
  - kinetic terms with vectors and antisymmetric tensors
  - topological terms (coupling the antisymmetric tensors)
  - Generalized Chern-Simons terms
- For electric gaugings: antisymmetric tensors decouple, no topological terms

Which antisymmetric tensors ?

- For action invariance $B_{\mu\nu\alpha}$:
  algebra closes modulo field equations: ‘on-shell algebra’
- Algebra simpler when one uses antisymmetric tensors $B_{\mu\nu}{}^{(MN)}$.
- Constraints restrict the (MN) combinations to a particular representation of the global symmetry group
- Hierarchy
  - One needs also higher order antisymmetric tensors, e.g. $C_{\mu\nu\rho}{}^{M(NP)}$.
  - Action can be constructed with embedding tensor considered as ‘spurious field’ : leads to a tensor hierarchy, where tensors appears as Lagrange multipliers for constraints on the embedding tensor.

B.de Wit, H. Nicolai, H. Samtleben, 2008;
E. Bergshoeff, J. Hartong, O. Hohm, M. Hübscher, T. Ortín, 2009

Algebra

- Jacobi relation modified due to symmetric parts of $X_{(MN)}{}^P$:

$$X_{[MN]}{}^P X_{[QP]}{}^R + X_{[QM]}{}^P X_{[NP]}{}^R + X_{[NQ]}{}^P X_{[MP]}{}^R$$
$$= -\tfrac{1}{3}\left(X_{[MN]}{}^P X_{(QP)}{}^R + X_{[QM]}{}^P X_{(NP)}{}^R + X_{[NQ]}{}^P X_{(MP)}{}^R\right) .$$

- These $X_{(MN)}{}^P$
  - define zero modes of the transformations (reducible algebra),
  - are considered as fields (soft algebra),
  - imply that the algebra needs field equations (open algebra).

⇒ all the features for which the field-antifield (Batalin-Vilkovisky) formalism was designed

Field-antifield formulation

- originally designed for quantisation: ghosts-antighosts + gauge fixing
- it is also useful for encoding all the relations of a general gauge theory in one *master equation*
- Basic ingredients:
  - classical fields completed by
  - any symmetry ! ghost
  - any zero mode ! ghost for ghost
  - …

For every Field !
9 Antifield

PS: antifields are **not** the antighosts, …

Fields and Antifields

- As canonical conjugates but with opposite statistics
- Antibrackets similar to Poisson brackets, but symmetric

Φ^A	stat	gh	Φ^*_A	stat	gh
ϕ^i	$+$	0	ϕ^*_i	$-$	-1
c^a	$-$	1	c^*_a	$+$	-2
c^{a_1}	$+$	2	$c^*_{a_1}$	$-$	-3
			$\ldots$		

$$(F,G) = F\frac{\overleftarrow{\partial}}{\partial\Phi^A}\cdot\frac{\overrightarrow{\partial}}{\partial\Phi^*_A}G - F\frac{\overleftarrow{\partial}}{\partial\Phi^*_A}\cdot\frac{\overrightarrow{\partial}}{\partial\Phi^A}G = (G,F)$$

- Extended action $S(\Phi,\Phi^*)$
  - classical limit $S(\Phi,0)=S_{cl}(\phi)$
  - master equation: $(S,S)=0$ or $(S,S)=2i$~ ΔS when 9 anomalies
  - properness condition

$$R_{\alpha\beta} \equiv \frac{\partial}{\partial z^\alpha}\frac{\partial}{\partial z^\beta}S \text{ has rank } N$$

$$z^\alpha = \{\Phi^A, \Phi^*_A\}, \qquad \alpha = 1,\ldots,2N$$

Expansion of extended action and master equation

$$\begin{aligned} S &= S_{cl}(\phi) + \boxed{\phi_i^* R^i{}_a c^a} \\ &\quad + \boxed{c_a^* \left(Z^a{}_{a_1} c^{a_1}\right.} + f^a{}_{bc} c^c c^b \Big) + \phi_i^* \phi_j^* \left(V^{ji}{}_{a_1} c^{a_1} + E^{ji}{}_{bc} c^c c^b\right) \\ &\quad + \boxed{c_{a_1}^* \left(Z^{a_1}{}_{a_2} c^{a_2}\right.} + A^{a_1}{}_{b_1 c} c^c c^{b_1} + F^{a_1}{}_{abc} c^c c^b c^a \Big) + \ldots \end{aligned}$$

- main terms that determine transformations, zero modes, … .
- if these are sufficiently non-singular (properness condition) existence of solution of the master equation is guaranteed.
- then master equation contains all the modified Jacobi identities and similar structure equations

Φ^A	stat	gh	Φ^*_A	stat	gh
ϕ^i	$+$	0	ϕ_i^*	$-$	-1
c^a	$-$	1	c_a^*	$+$	-2
c^{a_1}	$+$	2	$c_{a_1}^*$	$-$	-3
…					

Application in embedding tensor formalism

$$\phi^i = \left\{z,\, g_{\mu\nu},\, A^{\mu M},\, B^{\mu\nu\, MN},\, C^{\mu\nu\rho\, MNP},\ldots\right\}$$
$$c^a = \left\{c^M,\, c^{\mu MN},\, c^{\mu\nu\, MNP},\ldots\right\}$$
$$c^{a_1} = \left\{c^{MN}_{(1)},\, c^{\mu MNP}_{(1)},\, c^{MNPQ}_{(1)},\ldots\right\}$$
$$\ldots$$

F. Coomans, AVP, in preparation

$$H^{\mu\nu M} = F^{\mu\nu M} + X_{(NP)}{}^M B^{\mu\nu NP}.$$

$$Y^{NP}{}_{Q[RS]} = 2\left(\delta_Q^{[N} X_{(RS)}{}^{P]} - X_{Q[R}{}^{[N}\delta_{S]}^{P]}\right)$$

$$\delta A_\mu{}^M = \partial_\mu \Lambda^M + X_{PQ}{}^M A_\mu{}^P \Lambda^Q \equiv D_\mu \Lambda^M$$

In the extended action:

$$S = S_{cl} +$$
$$+A^*_{\mu M}(\qquad D^\mu c^M$$

$$\phi_i^* R^i{}_a c^a$$

$$c_a^* Z^a{}_{a_1} c^{a_1}$$

$$c_{a_1}^* Z^{a_1}{}_{a_2} c^{a_2}$$

Conclusions

- Embedding tensor formalism allows to describe gauged supergravities in a duality-covariant description
- Has open algebra, reducible algebra, … :
  all the features for which the field-antifield formalism (Batalin-Vilkovisky) is designed.
- Has hierarchy of fields, but also of zero modes, zero for zero modes, …
- Master equation produces all the structure equations of the algebra.
- Can also be the first step for quantisation
  (add trivial systems with antighosts
  + canonical transformation for gauge fixing)